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JANUARY 1959

The AMERICAN MATHEMATICAL MONTHLY

(Founded in 1894 by Benjamin F. Finkel)

Ralph D. James, *Editor*

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THE CLOCK PARADOX IN RELATIVITY THEORY*

ALFRED SCHILD, The University of Texas and Physics Laboratory, Hughes Aircraft Company

The clock or the twin effect of the theory of relativity is essentially this: Consider a pair of twins. Immediately after birth they are separated. One of them, the first one, remains on earth; the second one is put in a rocket ship and flown to Alpha Centauri at a pretty high speed, 99% that of light. Alpha Centauri is the nearest star; it is about four light years away from us. As soon as the second twin gets to Alpha Centauri, he turns around and flies back to earth at the same high speed. When the two twins meet again, the first one, the one who stayed behind on earth, will be *eight years old.* He will be eight years old in every sense of the word. For example, he will be able to talk quite well and read a little bit. He may have just finished second grade and be about to enter into third. The second twin, the one who took the journey, on his return will be approximately *one year old.* Again, he will be one year old in every sense of the word. He will still need diapers, he will be barely able to walk and he won't be able to talk much.

This effect is also known as the clock or the twin paradox and it is the paradox that we wish to discuss today. There are a few other problems which I am going to mention very briefly but which we are not going to discuss in any detail.

The first problem is that of experimental verification. Has the clock effect ever been observed? The answer is "yes." The fact that fast μ mesons, produced in our atmosphere by cosmic rays, decay more slowly than μ mesons at rest is a verification of what is essentially the clock or the twin effect.

The second problem is of a technological nature. Is it likely that in the near future we can let a human being take a trip of this kind and observe such a large effect? The answer is "no." The energy required to produce such huge speeds in a macroscopic body is completely prohibitive, and from the technological point of view an effect on the scale that we have considered is not going to be observed in the foreseeable future.

The third problem is of a sociological nature. If we consider a young man in his twenties going off on a faster and more extended trip than that of our twin,

* Presented to the Thirty-ninth Summer Meeting of the Mathematical Association of America, August 26, 1958. This talk was first given in January 1958 at a seminar at the Wright Air Development Center in Dayton, Ohio. It was also given in July 1958 at the Research Laboratories of the Hughes Aircraft Co. and at the RAND Corporation.

This expository paper, it is hoped, will help to dispel the confusion on the concept of time in relativity theory, which is still as wide-spread as it is unnecessary. It contains simple and basic ideas which were all known and understood as far back as 1905. Under the circumstances, it seemed unnecessary to give references to large quantities of publication, controversy and polemic which has appeared in the intervening years.

The writing of this paper was supported in part by U. S. Air Force Contract AF 33(616)-5832, monitored by the Aeronautical Research Laboratory, Wright Air Development Center.

we might ask whether upon his return to earth the man should or would marry his own great-great-granddaughter. Well, relativity doesn't say anything about that; it just says that he could.

The clock or the twin paradox exists on two levels.

The first is the layman's level. The clock or the twin effect runs counter to his usual concept of time. Here the only possible answer is that if such an effect ever becomes commonplace and becomes part of people's everyday experience, then the layman will simply have to get used to thinking about time in a different way. The situation is somewhat analogous to the concept of "up" and "down." In the history of the development of mankind and, in fact, in the development of each individual human being, at first there was an immediate generalization from experience of a small neighborhood on the surface of the earth; and this generalization led to the assumption that there is a uniform up and down direction which permeates the whole universe. I think every child, when it first hears that the earth is round, immediately asks, "Well, why don't people in Australia fall off?" Ultimately the child, and the human race as a whole, got used to the idea of gravitation and to the fact that there is no universal up and down direction throughout the universe. The clock or the twin effect says something very similar about time. There is no universal time that permeates the whole universe.

In this connection, we should perhaps mention that in classical Newtonian physics there is a universal time or, equivalently, that time is *integrable*. On the other hand, the special theory of relativity of 1905, the theory developed by Lorentz and Poincaré and Einstein, predicts that time in general is *non-integrable*. This is the twin effect. Now, both classical physics and the theory of relativity are perfectly self-consistent logically. It is not possible to decide between them by argument. The only difference is this: Special relativity agrees with facts, with experiments and with observation, and Newtonian theory does not.

At a higher level there is a more subtle paradox—that of the physicist who knows some relativity theory. The Lorentz transformation implies that a moving clock goes slow. And now the physicist argues as follows: The first twin, the one who stays behind on earth, says that the second twin's clock goes slow. Thus, when the two twins meet again, he argues, twin 1 should be *older* than twin 2. Well, he is correct in his conclusion. Now twin 2, during each of the two legs of his journey, can also apply the Lorentz transformation and say that the clock of the first twin goes slow, that time goes slow for the first twin, that he ages more slowly, and thus that when the two twins meet again, twin 1 should be *younger* than twin 2. This, of course, is paradoxical and, in fact, if correct, would mean that relativity theory is inconsistent. Actually, the argument of twin 2 is false, as we are going to show later on.

A good many physicists believe that this paradox can only be resolved by the general theory of relativity. They find great comfort in this, because they don't know any general relativity and feel that they don't have to worry about the problem until they decide to learn general relativity. However, they are

quite wrong. The twin effect, and certainly an effect of the large order of magnitude that we have been considering, is one of the special theory of relativity. The paradox can and must be resolved within the context of just the special theory of relativity. This is important for the following reasons:

The special theory of relativity is now well established by physical observation. It is not only the original experiments, such as those of Michelson and Morley, which confirm the special theory of relativity. By now a very large number of phenomena agree with the special theory of relativity and verify it. For example, the Dirac electron—the electron that obeys the relativistic Dirac equations which predict the spectrum of the hydrogen atom to very good accuracy, the fact that an electron has spin, the fact that the gyromagnetic ratio of an electron has an extra factor 2 which one does not get from a naive model of a rotating rigid charge, the fact that positrons exist, the fact that relativistic quantum electrodynamics can now predict the Lamb shift with an accuracy of one in ten million, the fact that $E=mc^2$ and that nuclear bombs are going off all over the place; all those facts, and many others, are confirmations of the special theory of relativity.

The situation is quite different with the general theory of relativity. The general theory of relativity, Einstein's theory of gravitation which assumes a curved space-time background, is an aesthetically beautiful structure but there is only very meager experimental verification. There are just the three famous effects which are barely outside the range of experimental error. It is quite possible that the general theory of relativity is completely on the wrong track. For example, if satellite red-shift experiments do not agree with the predictions of the general theory of relativity or if we should find that positrons accelerate upwards in the earth's gravitational field, contradicting the equivalence principle, then the general theory of relativity would have to be abandoned. However, the twin effect would still exist.

Before I discuss the clock paradox, I should like to give you a brief bird's-eye view of classical, pre-relativistic physics and of the special theory of relativity.

Figure 1 shows a sketch of the structure of classical physics as it existed around the year 1900. There were two great theories, mechanics as perfected by Galileo and Newton, and electromagnetism as perfected by Faraday and Maxwell. Most physical phenomena at that time were known to fit into one or the other of those two theories. For example, acoustics and thermodynamics (or, equivalently, statistical mechanics) were known to be special cases of the theory of mechanical systems. Optics was known to be the theory of a special class of electromagnetic phenomena.

In classical mechanics the main law is the Newtonian equation of motion, the equation that states that force equals mass times acceleration. Actually, for our purposes, we have to state Newton's law of motion more carefully. We shall say that there exists a method for measuring the displacement of a particle and for measuring the time, there exists a *frame of reference*, such that in that

frame of reference force equals mass times acceleration. Such a frame of reference is known as an *inertial frame*, or as a Galilean or Newtonian frame. As soon as the existence of such a frame of reference is postulated it immediately follows from the laws of mechanics that there are an infinite number of inertial frames between which the laws of mechanics cannot distinguish. In mechanics we have a principle of democracy; one inertial frame is as good as any other. The different inertial frames move relative to one another with uniform velocity of translation. Mathematically, the transformation from one inertial frame to another is known as a Galilean transformation; it is $t=t'$, $x=x'+vt'$, *etc.*

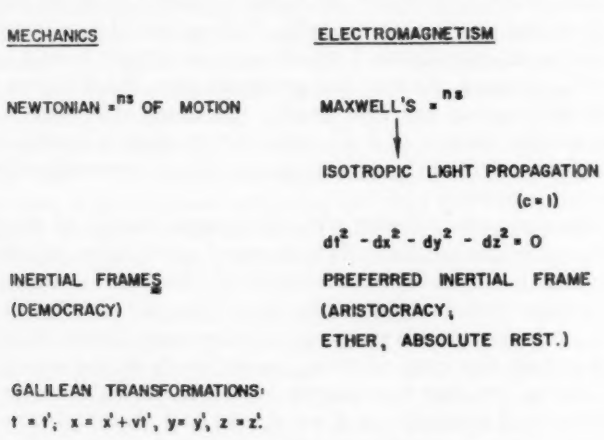

FIG. 1. Scheme of classical, pre-relativistic physics (~1900).

In electromagnetic theory the basic equations are Maxwell's equations. It is a direct mathematical consequence of Maxwell's equations that light propagates isotropically, with the same velocity in all directions. The velocity c of light is a constant that appears in Maxwell's equations. Throughout this talk we shall use units such that the velocity of light c is equal to one. This means that instead of using the centimeter as a unit of length, we use the light second or 300,000 kilometers as our unit of length. Then the equation $dt^2-dx^2-dy^2-dz^2=0$ expresses the law of isotropic light propagation. It is the equation which must be satisfied if a light pulse emitted at the point x, y, z at time t is to reach the point $x+dx$, $y+dy$, $z+dz$ at time $t+dt$. If it is now assumed that Maxwell's equations in their usual simple form are valid in one inertial frame of reference, then it was clear to the physicist of 1900 that Maxwell's equations could not hold in any other inertial frame. His argument went as follows: "Consider a light signal propagating to the right with velocity c and another light signal propagating to the left with velocity c. An observer moving to the right with the velocity v is catching up to the first light pulse, and for that observer

the first light pulse has a velocity smaller than c, the velocity $c-v$. On the other hand, the observer is running away from the second light pulse, and the second light pulse has for the observer a velocity greater than c, the velocity $c+v$. For the moving observer the two light pulses do not propagate with the same velocity. Thus for the moving observer Maxwell's equations cannot have the usual simple form. If they did, then again isotropic light propagation would follow as a mathematical consequence. It is therefore clear that in electromagnetic theory we have a principle of aristocracy. Of all the inertial frames which are possible in mechanics, electromagnetic theory picks out *one preferred frame.*" This preferred frame was called the ether or the frame of absolute rest.

The experiment of Michelson and Morley was designed to discover what this preferred frame was. More precisely, it was designed to measure the absolute velocity of the earth, the velocity of the earth through the ether. As you know, the Michelson-Morley experiment, and other experiments of the same type, all gave a null result. The empirical facts were not in agreement with the structure of physical theory as it existed in 1900.

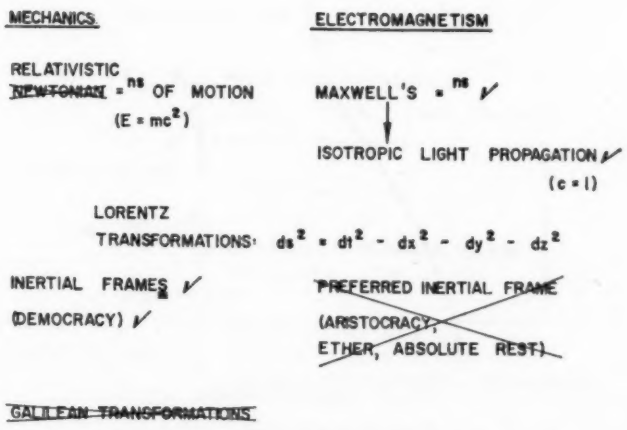

FIG. 2. Scheme of the special theory of relativity (1905).

Figure 2 shows how the special theory of relativity alters the scheme of classical physics in order to bring it into agreement with the new experimental facts. Maxwell's equations and their consequence, the isotropic propagation of light, are retained unchanged. So is the fact that we have democracy among inertial frames, that there are an infinite number of inertial frames moving relative to one another with uniform velocity, and that physical experiments are unable to distinguish between them. However, the Galilean transformations which do not leave Maxwell's equations invariant are thrown out. They are replaced by a new set of transformations, the *Lorentz transformations*, which are

those linear transformations which leave invariant the expression $ds^2=dt^2-dx^2-dy^2-dz^2$. In particular, the Lorentz transformations leave invariant the propagation properties of light which correspond to $ds^2=0$. It can be shown, although we shall not do this here, that in fact they leave invariant Maxwell's equations. However, the laws of mechanics, Newton's equations of motion, which were invariant under Galilean transformations, are not invariant under Lorentz transformations. Thus mechanics now seems to pick out a preferred inertial frame; an electromagnetic ether has simply been replaced by a mechanical ether. At this stage physicists were getting somewhat tired of the idea of an ether, whether electromagnetic or mechanic, and relativity theory makes mechanics Lorentz invariant by altering the laws of mechanics slightly, by replacing the Newtonian equations of motion by relativistic equations of motion. The famous formula $E=mc^2$ is a consequence of this change to relativistic equations of motion.

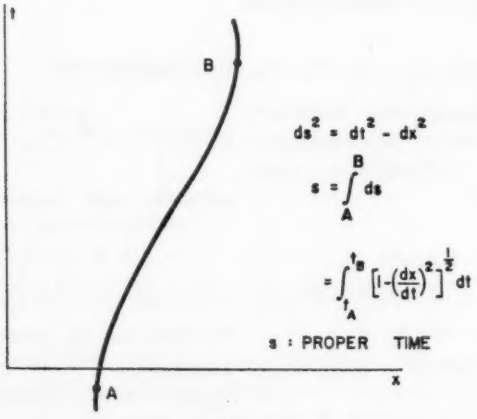

FIG. 3. Particle: World-line.

At this point, those in the audience who have never studied relativity theory should still have scruples, and I am sure they do. The argument about the moving observer catching up to one light pulse and running away from the other, with its conclusion that light cannot propagate isotropically for a moving observer, seems one of pure logic rather than physics. It is difficult to see how the formal scheme that I have presented avoids the difficulty. It was Einstein's great achievement to recognize that what is required here is to change our concepts about the physical behavior of time. In contrast to the situation in classical physics, we have to give up the notion of the absolute simultaneity of distant events, that is, that it has meaning independently of the observer to state that two events far apart in space happen at the same time. Also we have to give up

the notion that time is integrable; this, of course, is exactly the clock or the twin effect.

The expression for ds^2 that was used to characterize the Lorentz transformations has important physical and geometrical significance. In order to facilitate the discussion of this we shall reduce the number of dimensions of space from three to one by limiting ourselves to one-dimensional motions along the x-axis. Then $ds^2=dt^2-dx^2$.

Figure 3 shows a space-time representation of the motion of a particle. In such an xt-plane the motion of the particle is described graphically by a curve which we call the *world-line* of the particle. Here, and throughout our discussion, we shall limit ourselves to world-lines of slope greater than or equal to 1, corresponding to particle velocities less than or equal to the speed of light. Such

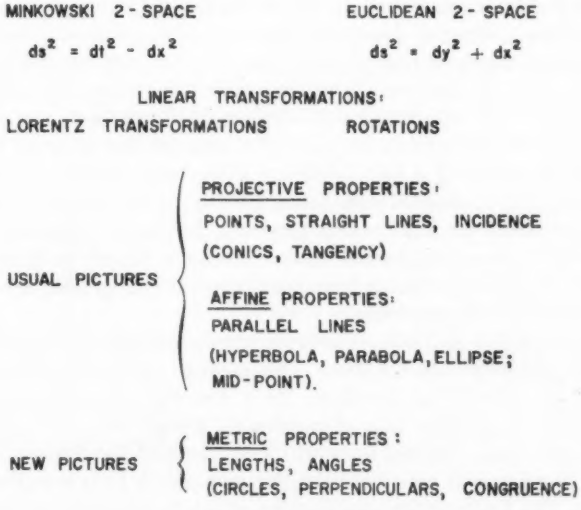

FIG. 4. Structure of geometry.

world-lines are called time-like, because the t-axis itself is of this type and is, in fact, the world-line of a particle at rest. Physically, the quantity s, equal to the integral of ds taken between two points A and B on the world-line of the particle, is the time interval between those two events as measured by the particle itself, by a clock carried by the particle and moving with it. This time interval s measured by physical clocks is called the *proper time* of the particle between the events A and B.

The geometrical significance of ds is illustrated in Figure 4. In space-time $ds^2=dt^2-dx^2$. We shall compare this to the expression $ds^2=dy^2+dx^2$, which measures the element of arc length ds in ordinary Euclidean two-space, where

x and y are rectangular Cartesian coordinates in the plane. The similarity of the two formulae suggests that space-time has many properties analogous to the ordinary plane, that $ds^2=dt^2-dx^2$ can be regarded as measuring the element of arc length ds in a new space, *Minkowski two-space.*

In order to discuss this, let us outline briefly the basic concepts of plane geometry. In geometry there are points and straight lines and properties of incidence, the fact that a point lies on a line or that a line goes through a point. As long as we restrict ourselves to these properties alone, we have *projective geometry.* An example of an axiom of projective geometry is that two points determine a unique line which is incident with the two points. It might appear that projective geometry, which uses only the primitive notions of points, straight lines, and incidence, would be a very feeble geometry with few interesting theorems. This is not so. Projective geometry is a very rich geometry. For example, the fact that a curve is a conic or that a line is tangent to a conic, is a purely projective property. However, in projective geometry we cannot distinguish different types of conics. If we go one step further and introduce the concept of parallelism, that two lines are parallel, then we have *affine geometry.* In affine geometry we can distinguish between the hyperbola, the parabola and the ellipse. We can define the mid-point of a straight line interval by the property that the diagonals of a parallelogram bisect one another. In fact, we can compare all lengths in parallel directions. Finally, if we introduce the concepts of distance and angle we have *metric geometry.* In metric geometry we can distinguish a circle from a noncircular conic, we can compare lengths in different directions, we can decide whether lines are perpendicular or not; in fact, we have the general notions of congruence.

As far as projective and affine concepts are concerned, Minkowski geometry and Euclidean geometry are identical. This means that in Minkowski geometry we can trust our Euclidean intuition, as it has been developed by our high school education; or another way of putting it, straight lines in space-time will look like straight lines, parallel lines will look like parallel lines, the mid-point of a straight line interval will look like the mid-point. It is not difficult to see why this is so. The Lorentz transformations which leave invariant the Minkowski element of arc $ds^2=dt^2-dx^2$ and the rotations which leave invariant the Euclidean element of arc $ds^2=dy^2+dx^2$ are all linear transformations. Now an arbitrary linear transformation preserves projective and affine properties; a point goes into a point, a straight line into a straight line, incidence is clearly preserved, and parallel lines go into parallel lines. However, metric properties are not preserved by an arbitrary linear transformation. Here the difference in sign between the Minkowski and Euclidean elements of arc becomes important. When it comes to metrical concepts, our Euclidean intuition is no longer valid in space-time—it cannot be trusted. Here we have to re-educate our intuition and learn to think in terms of new pictures. Thus, equal lengths in Minkowski geometry will not look equal, right angles will not look like right angles. For example, any straight line interval along the line $x=t$ has zero length and the

line $x=t$ is perpendicular to itself.

This then is a rough outline of the scheme of the special theory of relativity.

We are now prepared to discuss the clock or the twin effect in relativity theory. Figure 5 shows the world-lines of our two twins. Twin 1, who remained on earth, which we regard as being at rest in an inertial frame, has a straight world-line, whereas twin 2, who travelled to Alpha Centauri and returned, has the other world-line shown. In Euclidean geometry, two curves which join the same points A and B will in general not have the same length. Length in the Euclidean plane is nonintegrable. If one of the two curves happens to be a straight line, then we know that it is the shorter. The straight line is always shortest in Euclidean two-space; one side of a triangle is always smaller than the sum of the other two sides. In space-time, in Minkowski two-space, the situation is analogous. The lengths of the two world-lines between the events A and B will in general be different. Time is nonintegrable. The two twins will in general not have the same age when they are reunited at B. However, in Minkowski geometry the straight line is always the *longest*. Twin 1 will be older than twin 2 when they meet again.

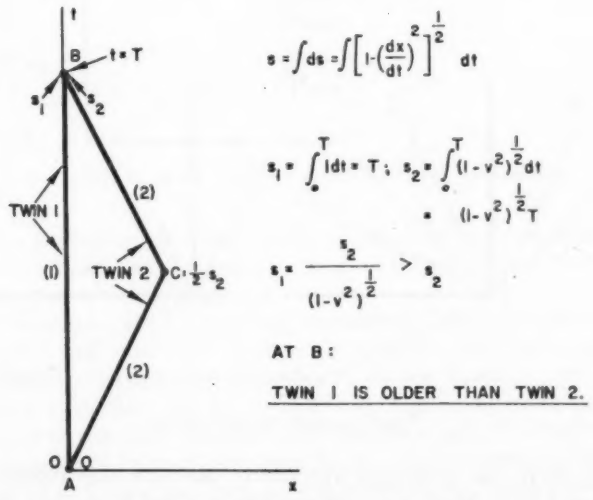

FIG. 5. The twin effect in special relativity.

This is not difficult to see. Let us choose a frame of reference in which twin 1 is at rest or, equivalently, a coordinate system in Minkowski two-space such that the t-axis falls along the line AB; and let us consider the integral for s in which the integrand is $(1-(dx/dt)^2)^{1/2}$. Along the world-line of twin 1 the integrand is one, whereas along the world-line of twin 2 the integrand is less than

one throughout. Therefore, the first integral is larger than the second. The first twin is older than the second twin when the two meet again; s_1 is greater than s_2. [In Euclidean geometry, the integrand in the formula for the arc length is $(1+(dx/dy)^2)^{1/2}$, the integration being with respect to y. This is why in Euclidean geometry the integrand for a curved line joining two points along the y-axis is throughout greater than one, and there it is the curved line which is longer than the straight line.]

Let us now consider a question which is often asked. Each of our two twins sees the other one first receding and then approaching. May one therefore conclude that the physical situation has perfect symmetry between the two twins and that one of them cannot be younger than the other when they meet again? The answer is "no." There is no symmetry between twin 1 and twin 2. Physically, twin 1 is throughout at rest in an inertial frame. Twin 1 never accelerates. Twin 2 changes inertial frames at the event C, at Alpha Centauri; twin 2 has

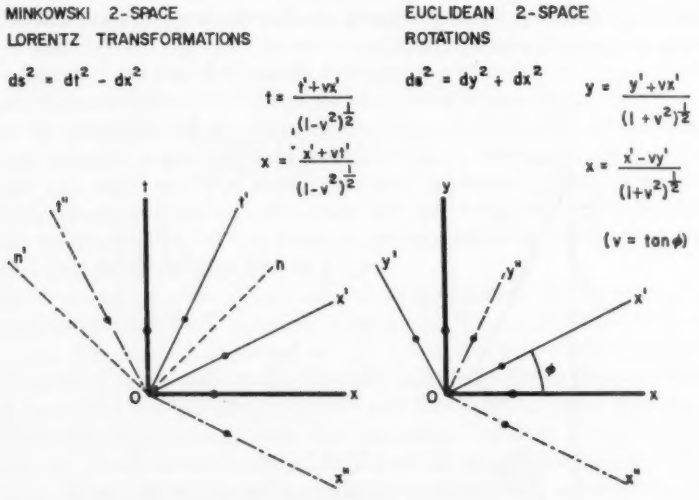

FIG. 6. Lorentz transformation.

to accelerate there. He is the one who has to use his rocket motors. In fact, if the acceleration is too high, he is the one who is going to be dead when the two twins meet again. Geometrically, the world-line of the first twin is a straight line, the world-line of the second twin is not—it has a corner at C.

Let us consider a situation which does have perfect symmetry between the two twins. Each of the two twins is now assumed at rest in an inertial frame. Twin 1 remains on earth, twin 2 shoots off in his rocket but keeps going. In this case the world-lines of the two twins are two straight lines. Now, two straight lines have the important property that they can intersect in one point but in

no other. The two twins can be together once, at birth, and there can compare their ages directly. But they can never do so again in the future. Thus, the whole problem as to which twin is younger and which twin is older disappears completely. While the two twins are flying apart it is a physically meaningless statement to say that one is younger or older than the other. For example, if twin 2, after reaching Alpha Centauri, decides that he prefers to be back on earth, reverses his motion and returns to earth, then twin 2 will be younger than twin 1 when they meet again. On the other hand, if twin 2 upon reaching Alpha Centauri decides that he likes travelling and keeps going, whereas twin 1 on earth finds after a while that he misses his brother, gets into a rocket ship that is even faster than his brother's and rushes off to catch up to twin 2, then when the two twins meet way out in space, twin 1 will be the younger one.

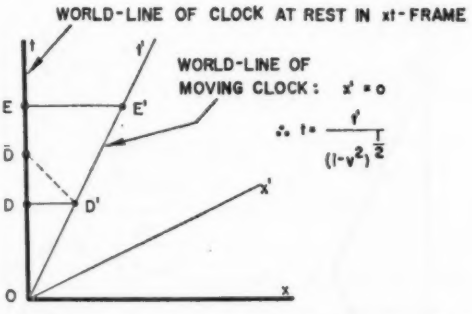

FIG. 7. Slowing down of moving clocks; Lorentz time dilatation.

This, then, is the complete story of the clock or the twin effect in the special theory of relativity. What remains is to clear up the paradox in connection with the Lorentz transformation argument.

Figure 6 illustrates the Lorentz transformations. They are simply the linear transformations that take one from a rectangular coordinate system xt to another rectangular coordinate system $x't'$. In the figure, x' and t' are perpendicular to one another, and so are x'' and t''. The heavy black dots shown on the axes are all at the same unit distance from the origin. The lines On and On' which are shown are each of zero length and each is perpendicular to itself. They are called *null lines* and they are the world-lines of light pulses travelling to the right and to the left, respectively. The Lorentz transformations of Minkowski space are the analogues of the rotations in Euclidean space. These are shown on the right. Note the similarity of the Lorentz transformation formulae to the formulae for rotations. Apart from the trivial change of name from t to y, the only difference is that of a few signs.

Consider now a clock, moving relative to the xt-frame, which is at rest at the origin of the $x't'$-frame so that its world-line has the equation $x' = 0$. Then the

Lorentz transformation formula for t takes on the simple form $t=t'/(1-v^2)^{1/2}$. This is illustrated in Figure 7. It is this formula which gives the slowing down of moving clocks by a factor $(1-v^2)^{1/2}$. We shall call it the *Lorentz time dilatation*. More precisely, the Lorentz time dilatation is this: Consider two clocks, one at rest in the xt-frame, with the t-axis as its world-line, the other moving with the t'-axis as its world-line. Events along the world-lines of the two clocks are brought into correspondence by using the relation of simultaneity with respect to the xt-frame, that is, by drawing lines parallel to the x-axis for which t =constant. D and D' are such a pair of corresponding events and so are E and E'. Then the length DE, which is a time interval measured by the rest clock, is greater by the Lorentz time dilatation factor $1/(1-v^2)^{1/2}$ than the length $D'E'$, which is a time interval measured by the moving clock.

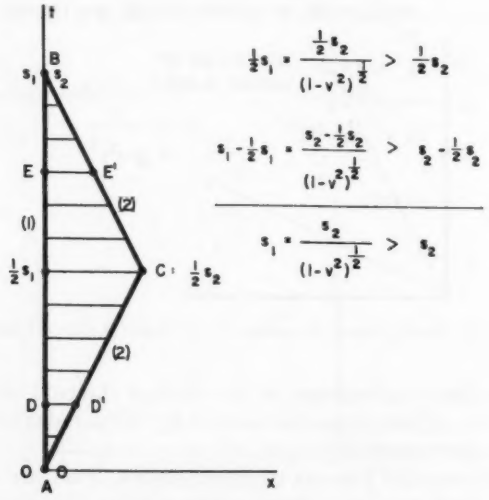

FIG. 8. Lorentz time dilatation argument; point of view of twin 1.

It is important to stress that the Lorentz time dilatation is not obtained from direct physical observation. It is really quite artificial and is based on a combination of physical observation and of mathematical calculation. An observer at rest in the xt-frame will at the event D be unaware of the event D' in the history of the moving clock. It is only later, at $\overline{D}$, that a light signal from D' reaches the observer at rest, and that he first sees the event D' of the moving clock. The observer must then sit down with pencil and paper, and make a mathematical calculation to correct for the time that light has taken to reach him, before he can identify the event D in his history with the event D' in the history of the moving clock.

Figure 8 illustrates the Lorentz time dilatation argument for the twin effect as given by twin 1. He uses simultaneity with respect to his xt-frame, lines parallel to the x-axis, to bring events into correspondence such as D with D' and E with E'. He then finds that when he and his brother are reunited, his age s_1 is related to the age s_2 of his brother by the Lorentz time dilatation formula, $s_1 = s_2/1-v^2)^{1/2}$. Thus s_1 is greater than s_2, and the first twin is older than the second when they meet again. This, of course, is the same result that we obtained previously.

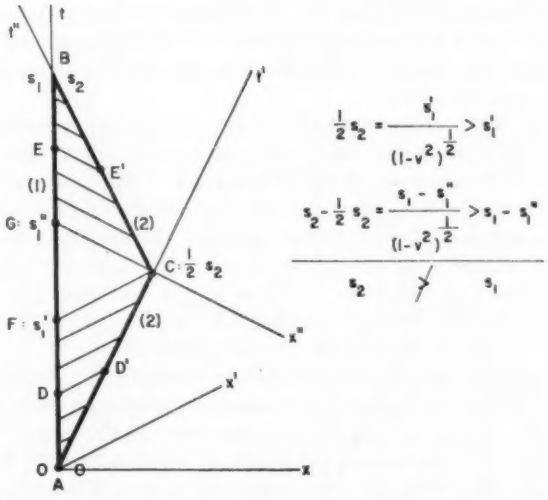

FIG. 9. Lorentz time dilatation argument; point of view of twin 2.

Figure 9 illustrates the Lorentz time dilatation argument as given by twin 2. On the first leg of the second twin's journey, from earth to Alpha Centauri, he can consider himself at rest in an inertial frame $x't'$. Twin 2 can use simultaneity relative to his frame, lines parallel to the x'-axis, to bring into correspondence events such as D' with D, and finally C with F. It is true that the length $AC = \frac{1}{2}s_2$ is greater than the length $AF = s_1'$ by the Lorentz time dilatation factor. On the second leg of the second twin's journey, from Alpha Centauri back to earth, twin 2 can again consider himself at rest in an inertial frame, the $x''t''$-frame. Again, he can use simultaneity, lines parallel to the x''-axis, to bring events into correspondence, such as C with G and E' with E. Again, the length $CB = s_2 - \frac{1}{2}s_2$ is greater than the length $GB = s_1 - s_1''$ by the Lorentz time dilatation factor. However, from the two inequalities twin 2 cannot conclude that s_2 is greater than s_1, that twin 2 is older than twin 1 when they meet again. This is because in the second twin's argument all the events from F to G in the history of the first twin have been left out.

Does this mean that at the event C twin 2 sees his brother suddenly and instantaneously changing from the age s_1' at F to the age s_1'' at G? The answer is "no." There is no physical effect that corresponds to a discontinuous jump in time in either twin's life. What has happened is simply this: At the event C, twin 2 suddenly and discontinuously changed his method of mathematical calculation, his method of correcting for the time that light takes to reach him. The only discontinuity is that he exchanged one method of calculation for another. Twin 2 physically sees all the events from F to G in the life of twin 1. His first method of calculation makes those events correspond to events in his own life which happen later than C. His second method of calculation makes those events correpond to events in his own life which happened before C. By suddenly changing his mathematics twin 2 misses the events from F to G—but he misses them only on paper.

This disposes of the fallacious Lorentz transformation argument and of the paradox. It is perhaps worth mentioning in passing that there is a simple Lorentz transformation argument which is correct: Twin 1 says that his brother travels with practically the speed of light for a distance of 4 light years and back; hence the journey takes $2\times 4=8$ years by the first twin's clock. Twin 2 also says that he travels with practically the speed of light relative to the earth; but for twin 2 the distance from earth to Alpha Centauri has undergone a Lorentz contraction; the distance has shrunk to $\frac{1}{2}$ of a light year; hence the round trip takes $2\times\frac{1}{2}=1$ year by the second twin's clock.

It is instructive to discuss what each twin actually sees by direct physical observation, by looking at his brother through a telescope. Figure 10, in the diagram on the right, illustrates what twin 1 observes. At the event $\overline{D}$ he observes twin 2 at the event D', the two points $\overline{D}$ and D' being related by the world-line of a light pulse moving to the left from twin 2 to twin 1. The correspondence of events $\overline{D}$ to D', $\overline{E}$ to E', $\overline{C}$ to C has immediate observational reality and is physically meaningful. From the event A to the event $\overline{C}$, twin 1 sees his brother recede. He sees his brother's clocks go slow, he sees his brother ageing more slowly. However, the time dilatation factor is not given by the artificial Lorentz time dilatation but is instead $[(1+v)/(1-v)]^{1/2}$. This formula is that of the relativistic *Doppler shift*, and we say that twin 1 observes a *red shift* in twin 2. This is because if twin 2 emits any spectral lines, then twin 1 will see these lines shifted towards the red. From the event $\overline{C}$ to the event B, twin 1 sees his brother approaching. The clocks of the second twin seem to be going fast, twin 2 is seen to be ageing more rapidly; we say that twin 1 observes a *violet shift*. The time dilatation factor is the reciprocal of the previous one. It is $[(1-v)/(1+v)]^{1/2}$. As seen by twin 2, in the diagram on the right, twin 1 is observed to be receding between the events A and C. The clocks of twin 1 seem to be going slow, twin 1 seems to be ageing more slowly; we say that twin 2 observes a red shift. Between the event C and the event B, twin 2 observes his brother approaching. The clocks of twin 1 are observed to go fast, twin 1 is observed to be ageing more rapidly; we say that twin 2 is observing a violet shift. The Doppler shift for-

mulae corresponding to these arguments are shown beneath the diagrams. By eliminating $\bar{s}_1$, the age of the first twin at the event $\overline{C}$, or by eliminating $\bar{\bar{s}}_1$, the age of twin 1 at the event $\overline{\overline{C}}$, we obtain in either case the old result that $s_1 = s_2/(1-v^2)^{1/2}$, that s_1 is greater than s_2, that twin 1 is older than twin 2 when the two twins meet at B. We see then that neither twin sees his brother ageing abruptly and discontinuously. Each twin only sees a jump in the rate of ageing, a transition from a red shift to a violet shift.

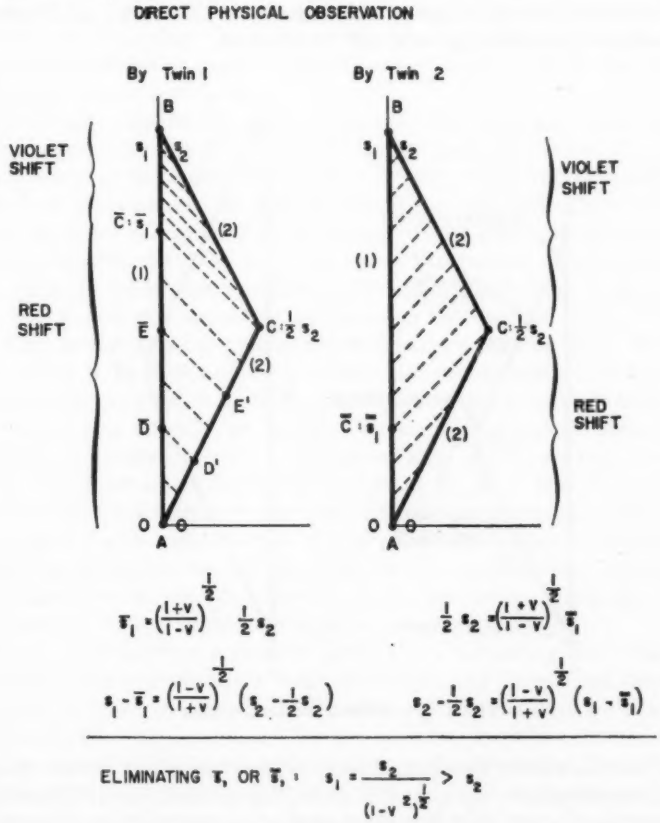

FIG. 10. Doppler shift argument.

The discontinuity in the rates at which clocks go is due to the fact that we have simplified our problem and put in a sharp corner at C; that is, we have assumed that twin 2 suffers an infinite acceleration there. Even the discontinuity in rates can be avoided by rounding out all corners of the space-time diagram.

This is shown below in Figure 11. At A twin 2 gradually accelerates to his final velocity, at C he gradually decelerates to rest and then accelerates until his velocity is reversed and, finally, at B he gradually decelerates to rest. This rounding out of corners can be done so smoothly that there is no discontinuity left at all. In direct observation each twin first sees his brother ageing at the same rate, then sees the rate gradually slowing down as a red shift develops, at a later time sees this red shift gradually changing to a violet shift, and finally near the end of the journey sees the violet shift gradually disappearing until the two brothers again age at the same rate. Of course, at B when the brothers are reunited their ages will still be different.

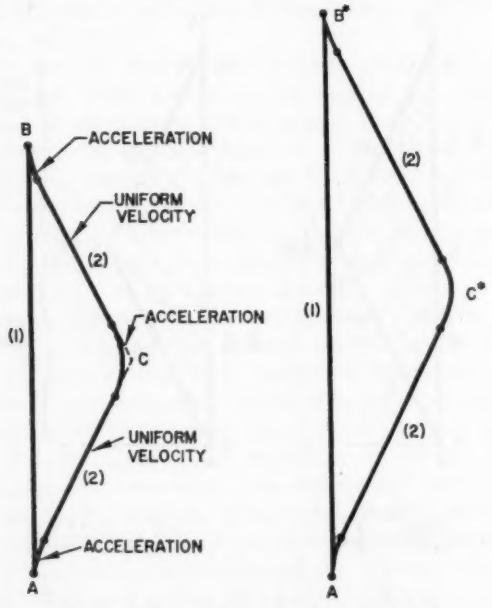

FIG. 11. Finite accelerations. Longer trip.

We shall now ask some simple and common questions and answer them.

First, is it reasonable to say that it is during his acceleration at C that twin 2 suddenly loses time or age, and that this loss at C causes the final age difference at B? The answer is "no." It is just as unreasonable to blame the acceleration at C for the total age difference of the twins as it would be, in the case of a triangle ACB in the ordinary Euclidean plane, to say that the larger length of the path ACB, as compared to the straight path AB, is caused by a sudden gain of length at the corner C.

The other question is this. During the periods of acceleration of twin 2, is it

possible that a new and unknown physical process influences the clocks and the ageing mechanism of twin 2 in such a way as always to compensate for the relativistic effect of age difference, so that the twin effect disappears? Again, the answer is "no." The total amount of the age difference is proportional to the overall size of the space-time diagram of the twins' world-lines. By enlarging the figure to AC^*B^*, as shown in the diagram on the left, we leave the accelerations unchanged but increase the total amount of the age difference. Thus, even if we assumed that an unknown physical effect during acceleration can compensate for a time difference of seven years in the trip of twin 2 to Alpha Centauri and back, then the same acclerations cannot compensate for an age difference of millions of years if twin 2 instead took a trip all the way out to a neighboring galaxy and back.

This brings us to the last topic of our talk. We have seen that any physical effect of acceleration on clocks cannot change the basic fact of the existence of the twin effect in the special theory of relativity. However, it is still legitimate to ask how physical clocks behave during acceleration. Do ordinary clocks actually measure the integral of ds during accelerated motion? In the limit of small accelerations, say for accelerations of the general order of magnitude of one g, we know from experience that physical clocks behave as idealized clocks do in theory, and that according to relativity theory they measure the integral of ds. This need not be the case when accelerations become large. If we take two clocks which have different mechanisms but which under ordinary terrestrial conditions measure the same time, then if those two clocks are both put in the same rocket ship there is no reason why we should expect them to continue measuring the same time when the rocket ship undergoes large accelerations. Clearly the forces which constrain the clocks to follow the accelerated motion may have different effects on the two different mechanisms. Here, as often in science, it is worth while to turn the question upside-down. Instead of asking whether usual clocks measure the integral of ds during large accelerations, let us rather ask how we have to construct clocks, how we have to design the physical measurement of time, in order to obtain the proper time $\int ds$.

Figure 12 illustrates a possible method for achieving this. The method is valid within the frameworks of both the special and the general theories of relativity. We consider the universe filled with a *dilute free clock gas*. By "clock gas" we mean that for any event and for any velocity there is a clock of the gas near the given event which moves with nearly the given velocity. By the word "free" we mean that each of the clocks is in free fall, that its world-line is a straight line in special relativity or a geodesic in general relativity. By the word "dilute" we mean that the clocks are spaced sufficiently far apart so that they do not influence one another's behavior, that, for example, their mutual gravitational attraction is negligible. We assume that all the clocks in the gas have identical construction, although the precise mechanism is not important. However, we do not assume that the clocks are synchronized, since in relativity theory the concept of absolute simultaneity of distant events is not valid. An observer in

general accelerated motion would use the dilute free clock gas as follows: During any small interval in the history of the observer, he uses that free clock of the gas which is tangent to his world-line to measure the small proper time interval elapsed. Note that the observer only uses a small time *difference* in the reading of the free clock, so that the absolute time shown by the clock is unimportant. The observer finally adds together all the small time intervals, each one read by a different free clock, and it is this sum (more precisely the limit of such a sum, an integral) which he calls the total proper time elapsed between any two events in his history.

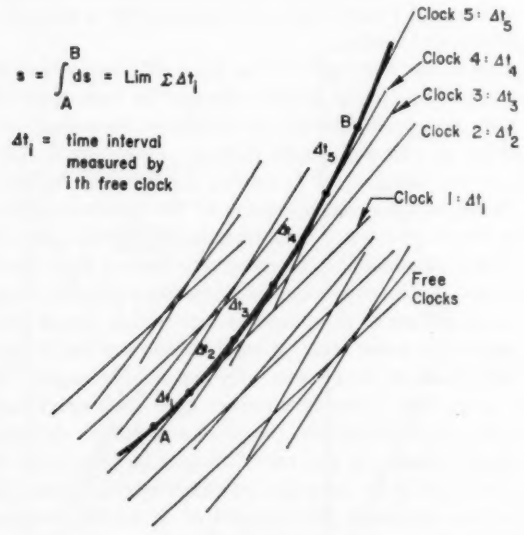
$s = \int_A^B ds = \text{Lim} \sum \Delta t_i$
Δt_i = time interval measured by i th free clock
Clock 5 : Δt_5
Clock 4 : Δt_4
Clock 3 : Δt_3
Clock 2 : Δt_2
Clock 1 : Δt_1
B
A
Free Clocks

FIG. 12. Dilute free clock gas.

We can say the same thing in a more dramatic fashion. The observer is in a rocket ship undergoing highly accelerated motion. As he looks out the window he sees one of the clocks of the gas sitting outside the window, at rest with respect to the rocket ship. He uses that clock to measure the passing of proper time. After a while, as the rocket ship accelerates, this clock drifts away and a new clock comes to rest outside the window. The observer in the rocket ship then switches to this new clock to measure elapsed proper time. He continues in this manner.

DIAGONAL ELEMENTS OF ORTHOGONAL MATRICES

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1. All numbers mentioned below are assumed to be real; and $N(x_1, \cdots, x_n)$ will denote the number of negative numbers among $x_1, \cdots, x_n$. By a *point* we shall mean an n-dimensional vector. An orthogonal matrix will be called *proper* or *improper* according as its determinant is $+1$ or -1. All matrices will be understood to be of type $n \times n$.

The following interesting result was established by A. Horn ([1] Theorem 8).

THEOREM 1. *The numbers $d_1, \cdots, d_n$ are the diagonal elements of a proper orthogonal matrix if and only if $(d_1, \cdots, d_n)$ lies in the convex envelope of those points of the form $(\pm 1, \cdots, \pm 1)$ which have an even number of negative coordinates.*

The object of the present note is to derive from this result an *effective* criterion for deciding whether n given numbers can be the diagonal elements of a proper orthogonal matrix. We shall, in fact, establish

THEOREM 2. *Necessary and sufficient conditions for the numbers $d_1, \cdots, d_n$ to be the diagonal elements of a proper orthogonal matrix are*

$$|d_j| \leqq 1 \qquad (j = 1, \cdots, n) \tag{1}$$

and

$$\sum_{k=1}^{n} |d_k| \leqq n - 2 + 2\lambda \min_{1 \leqq j \leqq n} |d_j|, \tag{2}$$

where λ is defined as 1 or 0 according as $N(d_1, \cdots, d_n)$ is even or odd.

Horn ([1] Theorem 9) obtained Theorem 2 for the case when the d's are nonnegative, and therefore for the case when $N(d_1, \cdots, d_n)$ is even; furthermore, the *necessity* of conditions (1) and (2) for all cases is implicit in his argument ([1] p. 627). However, the proof we shall give here depends on different ideas.

We note two consequences of Theorem 2.

COROLLARY 1. *Necessary and sufficient conditions for the numbers $d_1, \cdots, d_n$ to be the diagonal elements of an improper orthogonal matrix are* (1) *and*

$$\sum_{k=1}^{n} |d_k| \leqq n - 2 + 2(1 - \lambda) \min_{1 \leqq j \leqq n} |d_j|,$$

where λ is defined as in Theorem 2.

COROLLARY 2. *Necessary and sufficient conditions for the numbers $d_1, \cdots, d_n$ to be the diagonal elements of a proper orthogonal matrix and also of an improper orthogonal matrix are* (1) *and* $\sum_{k=1}^{n} |d_k| \leqq n-2$.

It is clear that Corollary 2 follows at once from Theorem 2 and Corollary 1, while Corollary 1 follows from Theorem 2 by virtue of the fact that $d_1, \cdots, d_n$ are the diagonal elements of an improper orthogonal matrix if and only if $-d_1, \cdots, -d_n$ are the diagonal elements of a proper orthogonal matrix.

I wish to express my thanks to the referee for his useful comments.

2. We shall need some preliminary results.

LEMMA 1. *For any given numbers $x_1, \cdots, x_n$ we have*:

(i) $$\max \sum_{k=1}^{n} \delta_k x_k = \sum_{k=1}^{n} |x_k|,$$

where the maximum is taken with respect to all sets of numbers $\delta_1, \cdots, \delta_n$ such that

$$\delta_1, \cdots, \delta_n = \pm 1, \qquad N(\delta_1, \cdots, \delta_n) \equiv N(x_1, \cdots, x_n) \pmod 2;$$

(ii) $$\max \sum_{k=1}^{n} \delta_k x_k = \sum_{k=1}^{n} |x_k| - 2 \min_{1 \leqq j \leqq n} |x_j|,$$

where the maximum is taken with respect to all sets of numbers $\delta_1, \cdots, \delta_n$ such that

(3) $$\delta_1, \cdots, \delta_n = \pm 1, \qquad N(\delta_1, \cdots, \delta_n) \not\equiv N(x_1, \cdots, x_n) \pmod 2.$$

In the first place, we have trivially

$$\sum_{k=1}^{n} \delta_k x_k \leqq \sum_{k=1}^{n} |x_k|.$$

Also, taking δ_k as $+1$ or -1 according as $x_k \geqq 0$ or $x_k < 0$, we obtain

$$\sum_{k=1}^{n} \delta_k x_k = \sum_{k=1}^{n} |x_k|.$$

This proves (i). Next, write $\min_{1 \leqq j \leqq n} |x_j| = |x_s|$. If $\delta_1, \cdots, \delta_n$ satisfy (3), then $\delta_i x_i \leqq 0$ for some i. Hence

$$\sum_{k=1}^{n} \delta_k x_k \leqq \sum_{k=1}^{n} |x_k| - 2|x_i| \leqq \sum_{k=1}^{n} |x_k| - 2|x_s|.$$

Again, take δ_s as $+1$ or -1 according as $x_s < 0$ or $x_s \geqq 0$ and, for $k \neq s$, take δ_k as $+1$ or -1 according as $x_k \geqq 0$ or $x_k < 0$. Then (3) is satisfied, and we have

$$\sum_{k=1}^{n} \delta_k x_k = \sum_{k=1}^{n} |x_k| - 2|x_s|.$$

Hence (ii) is proved.

LEMMA 2. *Let* $P=(x_1, \cdots, x_n)$ *and* $P_k=(x_{k1}, \cdots, x_{kn})$, $k=1, \cdots, m$, *be any points. Then* P *lies in the convex envelope of* $P_1, \cdots, P_m$ *if and only if, for any numbers* $a_1, \cdots, a_n$ *and some suffix* k *(which may depend on the* a*'s), we have*

$$a_1x_1 + \cdots + a_nx_n \leqq a_1x_{k1} + \cdots + a_nx_{kn}.$$

This result expresses the, intuitively obvious, fact that P lies in the convex envelope of $P_1, \cdots, P_m$ if and only if no hyperplane can separate P from all the points $P_1, \cdots, P_m$. For further details of this standard result, we refer to ([2] pp. 23–24).

As an immediate consequence of Theorem 1 and Lemma 2, we have

LEMMA 3. *The numbers* $d_1, \cdots, d_n$ *are the diagonal elements of a proper orthogonal matrix if and only if, for any numbers* $a_1, \cdots, a_n$, *we can find numbers* $\epsilon_1, \cdots, \epsilon_n$ *such that*

$$\epsilon_1, \cdots, \epsilon_n = \pm 1, \qquad N(\epsilon_1, \cdots, \epsilon_n) \equiv 0 \pmod 2 \tag{4}$$

and

$$\sum_{k=1}^{n} a_kd_k \leqq \sum_{k=1}^{n} a_k\epsilon_k. \tag{5}$$

3. We now come to the proof of Theorem 2. Assume, in the first place, that $d_1, \cdots, d_n$ are the diagonal elements of a proper orthogonal matrix. Then (1) is obviously satisfied. Take any numbers $a_1, \cdots, a_n$ such that

$$a_1, \cdots, a_n = \pm 1, \qquad N(a_1, \cdots, a_n) \equiv 1 \pmod 2. \tag{6}$$

Then, for suitable ϵ's satisfying (4), the relation (5) is valid. But $a_k\epsilon_k=-1$ for at least one value of k. Therefore $\sum_{k=1}^{n} a_kd_k \leqq n-2$, and so

$$\max \sum_{k=1}^{n} a_kd_k \leqq n-2, \tag{7}$$

where the maximum is taken with respect to all sets of numbers $a_1, \cdots, a_n$ satisfying (6). By Lemma 1, this maximum is equal to $\sum_{k=1}^{n} |d_k|$ if $N(d_1, \cdots, d_n)$ is odd and to $\sum_{k=1}^{n} |d_k| - 2 \min_{1\leq j\leq n} |d_j|$ if $N(d_1, \cdots, d_n)$ is even. Hence, in view of (7), (2) is satisfied. This proves the necessity of (1) and (2).

Next, assume that (1) and (2) are given. Let $a_1, \cdots, a_n$ be any numbers. By (1), we clearly have

$$\sum_{k=1}^{n} a_kd_k \leqq \sum_{k=1}^{n} |a_k|. \tag{8}$$

Now suppose that $N(a_1, \cdots, a_n)$ is odd and $N(d_1, \cdots, d_n)$ even. Then $a_id_i \leqq 0$ for some i; and therefore

$$\sum_{k=1}^{n} a_kd_k \leqq \sum_{k=1}^{n} |a_k|\,|d_k| - 2\lambda |a_i|\,|d_i|. \tag{9}$$

This inequality obviously also holds when $N(a_1, \cdots, a_n)$ and $N(d_1, \cdots, d_n)$ are both odd. Writing

$$\min_{1 \leqq j \leqq n} |a_j| = |a_r|, \qquad \min_{1 \leqq j \leqq n} |d_j| = |d_s|$$

and using (9), (1), and (2), we infer that, whenever $N(a_1, \cdots, a_n)$ is odd,

$$\begin{aligned}
\sum_{k=1}^{n} a_k d_k &\leqq \sum_{k=1}^{n} |a_k|\,|d_k| - 2\lambda |a_r|\,|d_s| \\
&= \sum_{k=1}^{n} |a_k| - \sum_{k=1}^{n} (1 - |d_k|)\,|a_k| - 2\lambda |a_r|\,|d_s| \\
&\leqq \sum_{k=1}^{n} |a_k| - \sum_{k=1}^{n} (1 - |d_k|)\,|a_r| - 2\lambda |a_r|\,|d_s| \\
&= \sum_{k=1}^{n} |a_k| - |a_r| \left(n - \sum_{k=1}^{n} |d_k| + 2\lambda |d_s| \right) \\
&\leqq \sum_{k=1}^{n} |a_k| - 2|a_r| .
\end{aligned}$$

Thus, in view of (8), we have for any values of $a_1, \cdots, a_n$:

$$\sum_{k=1}^{n} a_k d_k \leqq \begin{cases} \sum_{k=1}^{n} |a_k| & \text{when } N(a_1, \cdots, a_n) \text{ is even,} \\ \sum_{k=1}^{n} |a_k| - 2|a_r| & \text{when } N(a_1, \cdots, a_n) \text{ is odd.} \end{cases} \tag{10}$$

But, by Lemma 1, it is possible to choose numbers $\epsilon_1, \cdots, \epsilon_n$, subject to the conditions (4), such that the right-hand side in (10) is equal to $\sum_{k=1}^{n} a_k \epsilon_k$. Thus, given any numbers $a_1, \cdots, a_n$, it is possible to satisfy (4) and (5) simultaneously; and it follows by Lemma 3 that $d_1, \cdots, d_n$ are the diagonal elements of a proper orthogonal matrix.

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NEAR-RINGS

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1. Introduction. The ring structure of the endomorphisms of a commutative group is well known. Most of the ring properties do not hold for the endomorphisms of a noncommutative group. Closure with respect to the additive operation fails. Even in the closed system generated additively by the endomorphisms, commutativity of addition and the right distributive law do not hold, nor are there necessarily additive inverses. However, the system generated additively by the endomorphisms and anti-endomorphisms of a group form a group with respect to addition, a semigroup with respect to operator multiplication, and multiplication is left distributive with respect to addition. This system is an example of a near-ring.

A *near-ring* is an ordered triple $\mathfrak{P}=(P, +, \cdot)$, where P is a nonempty set, and $+$, $\cdot$ are binary compositions on P (called addition and multiplication) such that (i) $(P, +)$ is a group, (ii) $(P, \cdot)$ is a semigroup, and (iii) multiplication is left distributive with respect to addition. A ring is of course a near-ring. The set of all transformations from a group to itself relative to point-wise addition and operator multiplication is also a near-ring. Other examples are presented in Section 2.

A *near-field*, a near-ring in which every nonzero element has a multiplicative inverse, is an example which is useful in the study of certain non-Desarguesian planes [7, 9]. Near-rings appear to have application to the study of nonlinear operators, since the collection of all operators on a vector space forms a near-ring, and also appear to have application in characterizing endomorphisms of a group. Dickson [6] was the first to study finite near-fields. Zassenhaus [13] determined all finite near-fields while Kalscheuer [8] classified all near-fields over the reals. Neumann [10] considered near-fields connected with free groups. Blackett [2] considered a class of simple and semisimple near-rings, and Deskins [5] considered near-ring radicals. Other papers are also listed in the bibliography.

This paper discusses some of the elementary properties of near-rings. The notation and definitions are chosen so that ring theory is a special case. In Section 2 examples of near-rings are given which are used to illustrate the theory. The arithmetic of near-rings is discussed in Section 3. Certain special classes of near-rings pertinent to the development of the theory are presented in Section 4. In Section 5 it is shown that any near-ring is decomposable into a direct sum of special sub-near-rings, and the Peirce decomposition for rings is generalized. Extensions and embeddings are discussed in Section 6. In particular, the well-known connection between integral domains and fields is generalized to near-rings. The theory of ideals and homomorphisms is developed in Section 7 and the ideals of special classes of near-rings is characterized in Section 8. Section 9 indicates some problems and extensions of interest.

2. Examples. Let $\mathfrak{G}=(G, +)$ be a group. Let $\mathfrak{T}(\mathfrak{G})=(T(G), +, \cdot)$, where $T(G)$ is the set of all transformations on G, and $+, \cdot$ are the binary compositions on $T(G)$ defined by $g(\alpha+\beta)=g\alpha+g\beta$, $g(\alpha\cdot\beta)=(g\alpha)\beta$ (operator multiplication), $g\in G$. $\mathfrak{T}(\mathfrak{G})$ is a near-ring. The identity of $(T(G), +)$ is the transformation ζ which carries every element of G into O. The additive inverse, $-\alpha$, of α is $\alpha\rho$, where ρ is the anti-isomorphism of $\mathfrak{G}$ which maps every element of G into its inverse. The identity transformation ϵ is the multiplicative identity of $T(G)$ and $-\epsilon=\rho$.

Let $\mathfrak{T}_R(\mathfrak{G})=(T_R(G), +, \cdot)$ be the smallest near-ring in $\mathfrak{T}(\mathfrak{G})$ containing the endomorphisms of $\mathfrak{G}$. All the anti-endomorphisms of $\mathfrak{G}$ lie in $T_R(G)$ since $-\alpha=\alpha\rho$ is an anti-isomorphism of $\mathfrak{G}$ if α is an endomorphism. Every element of $T_R(G)$ is a sum of elements $\alpha_1+\cdots+\alpha_n$ where either α_i or $-\alpha_i$ is an endomorphism, $i=1, \cdots, n$. Since the endomorphisms are right distributive, $T_R(G)$ has the property that every element is a sum of right or "anti"-right distributive elements.

Let $\mathfrak{T}_Z(\mathfrak{G})=(T_Z(G), +, \cdot)$, where $T_Z(G)=\{\xi_g\in T(G) \mid g'\xi_g=g;\ g,\ g'\in G\}$. $\mathfrak{T}_Z(\mathfrak{G})$ is a sub-near-ring of $\mathfrak{T}(\mathfrak{G})$ which has the property that $\alpha\xi=\xi$ for every $\alpha\in T(G), \xi\in T_Z(G)$. It is the maximal sub-near-ring of $\mathfrak{T}(\mathfrak{G})$ with this property.

Let $\mathfrak{T}_C(\mathfrak{G})=(T_C(G), +, \cdot)$, where $T_C(G)=\{\alpha\in T(G) \mid 0\alpha=0\}$. $\mathfrak{T}_C(\mathfrak{G})$ is a sub-near-ring of $\mathfrak{T}(\mathfrak{G})$ with the property that $\zeta\alpha=\alpha\zeta$ for all $\alpha\in T_C(G)$. It is the maximal sub-near-ring of $\mathfrak{T}(\mathfrak{G})$ with this property.

If $\mathfrak{G}$ is a topological group, and $K(G)\subset T(G)$ the continuous transformations, then $(K(G), +, \cdot)$ is another sub-near-ring of $\mathfrak{T}(\mathfrak{G})$.

Other near-rings can be constructed from $\mathfrak{G}$. For example, define multiplication in G by $g\cdot h=h$, or $g\cdot h=0$, $g, h\in G$. Then $(G, +, \cdot)$ is a near-ring.

The analytic functions on the complex plane form a near-ring (with the right distributive law instead of the left distributive law when the usual functional notation is used) relative to point-wise addition and substitution, and another near-ring relative to point-wise multiplication and substitution.

Other examples of near-rings can be found in [3, 4].

3. The arithmetic of near-rings. The arithmetic of near-rings differs from the arithmetic of rings since the right distributive law and commutative law of addition are missing. For example, the zero of a near-ring may behave multiplicatively in a different fashion from the zero of a ring: $a0=0$, but $0a$ need not equal 0. Indeed $\zeta\xi=\xi$ for every $\xi\in T_Z(G)$. All the elements of $T_Z(G)$ behave multiplicatively on the right like zero (*i.e.* $\alpha\xi=\xi$, $\alpha\in T(G)$, $\xi\in T_Z(G)$).

Seemingly obvious properties of rings do not hold for near-rings. For example, even if a near-ring contains a multiplicative identity 1, its additive inverse -1 need not commute with all the elements. Multiplicative inverses, if they exist, may also behave in a bizarre manner. For example, commutative elements may have multiplicative inverses which do not commute. A study of the arithmetic of near-rings emphasizes the role played by the right distributive law and commutative addition in ring theory.

4. Special classes of near-rings. In this section restrictions on near-rings are introduced and some of the implications of these restrictions are indicated. A definition is needed. A *sub-near-ring* $\mathfrak{Q}$ of a near-ring $\mathfrak{P}=(P, +, \cdot)$ is a system $(Q, +, \cdot)$ such that $(Q, +)$ is a subgroup of $(P, +)$, and $(Q, \cdot)$ is a sub-semi-group of $(P, \cdot)$.

(a) *Rings.* Of course the most widely-studied near-rings are rings. A ring is a near-ring $\mathfrak{R}=(R, +, \cdot)$ with the two restrictions (i) $+$ is commutative, and (ii) the right distributive law holds. It is not sufficient to assume only (ii), as is seen by considering $(G, +, \cdot)$ in which $(G, +)$ is a noncommutative group and $g \cdot h=0$, $g, h \in G$.

If a near-ring $\mathfrak{P}$ *satisfies* (ii), *and* (iii) *every element of* P *is the product of two elements, then* $\mathfrak{P}$ *is a ring.* This result, first published by Olga Taussky, is easily proved by setting $a=a_1a_2$, $b=b_1b_2$ and considering the two expansions of $(a_1+b_1)(b_2+a_2)$. It follows that a near-ring with a multiplicative identity and satisfying (ii) is a ring. Thus the commutative law of addition for a ring with identity is redundant. There are other simple properties which guarantee (i) and (ii). For example, a near-ring $\mathfrak{P}$ having commutative multiplication and satisfying (iii) (*e.g.* P contains an identity) is a ring.

(b) *C-rings.* A *C-ring* is a near-ring $\mathfrak{C}$ with the property that $0c=0$ for every $c \in C$. A study of semi-simple C-rings was made by Blackett [2]. Rings and near-fields are C-rings. Other examples are $\mathfrak{T}_R(\mathfrak{G})$ and $\mathfrak{T}_C(\mathfrak{G})$ of Section 2. *A near-ring* $\mathfrak{P}$ *is a C-ring if and only if* $az=z$, $a \in P$ *implies that* $z=0$. For, if $\mathfrak{P}$ is a C-ring, assume that $az=z$ for every $a \in P$, then $0z=z$. But $0z=0$, so that $z=0$. On the other hand, if $az=z$, $a \in P$ implies that $z=0$, consider $0a$, $a \in P$. Then $b(0a)=(b0)a=0a$, $b \in P$, so that $0a=0$.

Let $\mathfrak{P}$ be a near-ring and $P_C=\{a \in P \mid 0a=0\}$. It is immediate that P_C determines a sub-near-ring of $\mathfrak{P}$ which is a maximal C-ring of P. Other characterizations of C-rings will be given in terms of ideals in Section 7.

(c) *Z-rings.* A *Z-ring* is a near-ring $\mathfrak{Z}$ with the property that $ab=b$ for every $a, b \in Z$. $\mathfrak{T}_Z(\mathfrak{G})$ defined in Section 2 is an example of a Z-ring. The following statements are immediate. Let $\mathfrak{G}=(G, +)$ be a group, and $\cdot$ the binary composition on G defined by $g \cdot h=h$, $g, h \in G$. Then the system $(G, +, \cdot)$ is a Z-ring, and conversely every Z-ring is characterized by a group $\mathfrak{G}$ and this multiplication. The system $(H, +, \cdot)$, where $\mathfrak{H}$ is a sub-group of $\mathfrak{G}$, is a Z-ring and a sub-near-ring of $(G, +, \cdot)$, and conversely.

Let $\mathfrak{P}$ be a near-ring and $P_Z=\{z \in P \mid az=z, a \in P\}$. The set P_Z determines a sub-near-ring of $\mathfrak{P}$ which is a Z-ring. Further, if $\mathfrak{Q}$ is a sub-near-ring of $\mathfrak{P}$ which is a Z-ring, then $\mathfrak{Q} \subset \mathfrak{P}_Z$. For, it is clear that $\mathfrak{P}_Z$ is a sub-near-ring of $\mathfrak{P}$ and a Z-ring. Let $\mathfrak{Q}$ be a sub-near-ring of $\mathfrak{P}$ which is a Z-ring. Then, since $0 \in Q$, $aq=a(0q)=(a0)q=0q=q$, for $q \in Q$, $a \in P$. Hence $q \in P_Z$ by definition.

It will be shown in Section 5 that every near-ring is a direct sum of a C-ring and a Z-ring.

(d) *R-rings.* An element $a \in P$ is *anti-right distributive* if $(b+c)a=ca+ba$; $b, c \in P$. It follows at once that (i) an element a is right distributive if and only

if $-a$ is anti-right distributive, (ii) the product ab is right distributive if a, b are both right distributive or both anti-right distributive, and (iii) the product ab is anti-right distributive if one of a, b is right distributive and the other anti-right distributive. The arithmetic of right and anti-right distributive elements closely approximates the arithmetic in rings. For example, if $\mathfrak{P}$ is a C-ring and Q the set of right and anti-right distributive elements of P, then for $p\in P$, $q\in Q$, $-(pq)=(-p)q=p(-q)$. The proof follows from the observation that $0=0q=(p+(-p))q=pq+(-p)q$ or $(-p)q+pq$.

An element $a\in P$ is *weakly right distributive* if it is a finite sum of right and anti-right distributive elements. An *R-ring* is a near-ring in which every element is weakly right distributive. *The set of weakly right distributive elements of a near-ring* $\mathfrak{P}$ *form a sub-near-ring* $\mathfrak{P}_R$. For, the sum and difference of weakly right distributive elements is clearly weakly right distributive, and it is easy to check that the product is also.

The smallest near-ring of transformations on a group $\mathfrak{G}$ containing the endomorphisms of $\mathfrak{G}$ is an R-ring ($\mathfrak{T}_R(\mathfrak{G})$ in Section 2). It is universal in the sense that every R-ring can be embedded isomorphically into such a weakly right distributive near-ring of transformations (see Section 6).

If addition is commutative in an R-ring it follows trivially that it is a ring.

(e) *M-rings.* An *M-ring* is a near-ring with commutative multiplication. Since 0 commutes with every element of an M-ring it follows that an M-ring is a C-ring. A condition that an M-ring be a ring was given in (a).

Let $P_M=\{a\in P \mid ap=pa,\ p\in P\}$ be the *center* of P. If $c\in P_M$, $b\in P$, then $b(-c)=(-b)c=-(bc)$, and the elements of P_M are right distributive relative to P. (*i.e.* $(a_1+a_2)c=a_1c+a_2c$; $a_1, a_2\in P\cdot c\in P_M$). The statement follows from the identities $b(-c)=-(bc)=-(cb)=c(-b)=(-b)c$ and $(a_1+a_2)c=c(a_1+a_2)=ca_1+ca_2=a_1c+a_2c$. Note that if $c\in P_M$, then $-c$ is anti-right distributive over P.

The center of a near-ring may be empty. It does not follow that if P_M is nonempty that its elements determine a near-ring. For example in $\mathfrak{T}(\mathfrak{G})$, the identity ϵ is in the center but the additive zero ζ is not. Hence a near-ring is not determined. However, if $\mathfrak{P}_M=(P_M, +, \cdot)$ is a sub-near-ring of $\mathfrak{P}$, then $\mathfrak{P}_M$ is a C-ring and the elements of P are right distributive over P_M. (*i.e.* $(c_1+c_2)a=c_1a+c_2a$; $c_1, c_2\in P_M$, $a\in P$.) Further, if every element of P_M can be expressed as the product of two elements of P_M, then P_M is a ring. For, if P_M is a near-ring, $O\in P_M$, and hence P is a C-ring. The identity $(c_1+c_2)a=a(c_1+c_2)=ac_1+ac_2=c_1a+c_2a$; $c_1, c_2\in P_M$, $a\in P$ shows that P is right distributive over P_M. The ring property follows from the previous remarks.

(f) *D-rings.* A *D-ring* is a near-ring $\mathfrak{D}$ such that (i) $ab=0$; $a, b\in D$ implies that either $a=0$ or $b=0$, and (ii) for every $a\in D$ there are nonzero elements $a_l, a_r\in D$ such that a_la and aa_r are in the multiplicative center D_M. Near-fields are D-rings. Integral domains are examples of commutative rings which are D-rings, and an example of a noncommutative D-ring is the ring of integral quaternions. The D-rings are generalizations of integral domains. It is shown in

Section 6 that D-rings can be embedded isomorphically into near-fields.

The identities $(0a)a_r = 0(aa_r) = (aa_r)0 = 0$ imply that $0a = 0$, so that *a D-ring is a C-ring. The condition that* $ab = 0$ *in a D-ring implies* $a = 0$ *or* $b = 0$ *may be replaced by the left and right cancellation laws, i.e.* $ab = ac$ *or* $ba = ca$ *implies* $b = c$ *if* $a \neq 0$. Assume $ab = ac$, $a \neq 0$, then $a(b-c) = 0$ implies $b - c = 0$. If $ba = ca$, $a \neq 0$, then $(ba)a_r = (ca)a_r$ so that $aa_r(b-c) = 0$. But $aa_r \neq 0$ implies $b = c$. The converse is immediate. Just as in field theory, *either a D-ring* $\mathfrak{D}$ *has characteristic* $p \neq 0$ (*prime*) *and contains an isomorphic image of* $GF(p)$, *or* $\mathfrak{D}$ *has characteristic* 0 *and contains an isomorphic image of the ring of integers. In the latter case if* $\mathfrak{D}$ *is a near-field it contains an isomorphic image of the field of rationals.* This statement is proved in exactly the same way as the corresponding theorem for fields. In case D has finite cardinality the cancellation law implies the unique solution of the equations $ax = b$ and $ya = b$, $a \neq 0$, which in turn implies that *a finite D-ring is a near-field.*

(g) *A-rings.* An *A-ring* is a near-ring with commutative addition. A ring is of course an A-ring. The near-ring $\mathfrak{T}(\mathfrak{G})$, where $\mathfrak{G}$ is a commutative group, is an example of an A-ring which is not a ring. However, an A-ring which is an R-ring is a ring. It is immediate that *a near-ring with a multiplicative identity is an A-ring if and only if* -1 *is in the multiplicative center.*

5. A decomposition theorem. Let $\mathfrak{P} = (P, +, \cdot)$ be a near-ring. The near-ring $\mathfrak{P}$ is the direct sum $\mathfrak{Q} \oplus \mathfrak{R}$ of the sub-near-rings $\mathfrak{Q}$ and $\mathfrak{R}$ if every element of P can be expressed as a unique sum of elements $q+r$; $q \in Q$, $r \in R$. It is clear that, given any two near-rings $\mathfrak{Q}$, $\mathfrak{R}$, a near-ring $\mathfrak{P}$ can be formed which is the direct sum of $\mathfrak{Q}$ and $\mathfrak{R}$ by defining coordinate-wise addition and multiplication on $Q \times R$. The near-ring $\mathfrak{P}$ has similar properties to the corresponding direct sum of two rings.

An *idempotent* is an element $e \in P$ such that $e^2 = e$. The following theorem generalizes the Peirce decomposition for rings.

THEOREM. *Let* e *be an idempotent in* P. *Then every element* $p \in P$ *has two unique decompositions* $p = ep + (-ep + p) = (p - ep) + ep$. *Thus* $\mathfrak{P} = \mathfrak{R} \oplus \mathfrak{S} = \mathfrak{S} \oplus \mathfrak{R}$, *where* $R = \{ep \mid p \in P\}$ *and* $S = \{s \in P \mid es = 0\}$.

The system $\mathfrak{R} = (R, +, \cdot)$ and $\mathfrak{S} = (S, +, \cdot)$ are near-rings. The elements $p - ep$ and $-ep + p$ are in S. Suppose $p = r_1 + s_1 = r_2 + s_2$. Then $-r_2 + r_1 = s_2 - s_1$ must be in $T = R \cap S$. But the only element in T is 0, for suppose $t \in T$. Then $et = 0$ and $t = ep$ for some $p \in P$, and $0 = et = e(ep) = e^2p = ep = t$. Thus $r_1 = r_2$ and $s_1 = s_2$. The uniqueness of the other representation is proved in the same way. Since $z^2 = z$ and $zp \in Z$ for $z \in P_z$, $p \in P$, the following corollary is immediate.

COROLLARY. *Every element* $z \in P_z$ *is an idempotent and determines the direct sum decompositions* $\mathfrak{P} = \mathfrak{P}_z \oplus \mathfrak{Q}_z = \mathfrak{Q}_z \oplus \mathfrak{P}_z$ *where* $Q_z = \{p \in P \mid zp = 0\}$.

The special case $z = 0$ is of interest since it shows that every near-ring can be expressed as the direct sum of its maximal sub-Z-ring $\mathfrak{P}_z$ (4c), and its maximal sub-C-ring $\mathfrak{P}_C$ (4b).

COROLLARY. *Every element* $p \in P$ *has unique decompositions* $p = p_1 + p_2 = p_2 + p_1'$ *where* $p_1, p_1' \in P_C$, *and* $p_2 \in P_Z$, *i.e.* $\mathfrak{P} = \mathfrak{P}_C \oplus \mathfrak{P}_Z = \mathfrak{P}_Z \oplus \mathfrak{P}_C$.

6. Extensions and embedding theorems. The well-known theorem on the embedding of a ring into the ring of endomorphisms of a commutative group can be generalized to near-rings. In fact *every near-ring can be embedded isomorphically into a near-ring of transformations* $\mathfrak{T}(\mathfrak{G})$ *on some group* $\mathfrak{G}$. *The elements in the isomorphic image can be taken as right multiplication operators on the additive group of the near-ring.* It follows that *every near-ring can be embedded isomorphically into a near-ring with identity.* These results are proved in [1]. In addition it is shown that *an R-ring (C-ring, Z-ring) can be embedded isomorphically into the R-ring (C-ring, Z-ring) of transformations* $\mathfrak{T}_R(\mathfrak{G})(\mathfrak{T}_C(\mathfrak{G}), \mathfrak{T}_Z(\mathfrak{G}))$ *on a group* $\mathfrak{G}$. *The elements of the image can again be taken as right multiplication operators.*

The theorem that an integral domain can be embedded in a field is now generalized to the embedding of D-rings into near-fields. It can be shown that it is a special case of a theorem proved for more general systems by Ore [12]. The theorem proved below includes as a special case the embedding of certain non-commutative integral domains into division rings. For example, the ring of integral quaternions, which is a D-ring, is embedded in the near-field of rational quaternions.

THEOREM. *A D-ring is isomorphic to a sub-near-ring of a near-field.*

LEMMA. *A D-ring* $\mathfrak{D}$ *can be embedded isomorphically in a D-ring* $\mathfrak{D}'$ *such that the nonzero elements of* D_M' *(its multiplicative center) form a group relative to multiplication.*

Consider the collection of pairs $(c, d) \in D_M \times D$ in which $c \neq 0$. This set may be partitioned into equivalence classes by defining $(c_1, d_1) \sim (c_2, d_2)$ whenever $c_1 d_2 = c_2 d_1$. It is easy to verify that $\sim$ is an equivalence relation. Let $[c, d]$ denote the equivalence class containing (c, d), and set $D' = \{[c, d], c \in D_M, c \neq 0, d \in D\}$. Let $+$, $\cdot$ be the binary compositions on D' defined by $[c_1, d_1] + [c_2, d_2] = [c_1 c_2, c_2 d_1 + c_1 d_2]$ and $[c_1, d_1] \cdot [c_2, d_2] = [c_1 c_2, d_1 d_2]$. Straightforward calculations show that the system $\mathfrak{D}' = (D', +, \cdot)$ is a near-ring. It is a D-ring, for if $[c_1, d_1] \cdot [c_2, d_2] = [c, 0]$ (the zero of D') then $c_1 c_2 0 = c d_1 d_2$, and either d_1 or d_2 is zero. This implies that either $[c_1, d_1]$ or $[c_2, d_2]$ is the zero element of D'. Further $[c, d_l]$ and $[c, d_r]$ have the property that $[c, d_l] \cdot [c, d]$ and $[c, d] \cdot [c, d_r]$ are in D_M'.

Let μ be the mapping from D to D' defined by $d\mu = [c, cd]$, $c \in D_M$. The mapping is clearly well defined and a simple calculation shows that it is an isomorphism. Thus $\mathfrak{D}'$ contains an isomorphic image of $\mathfrak{D}$.

The nonzero elements of D_M' are of the form $[c_1, c_2]$, $c_1, c_2 \in D_M$, $c_1, c_2 \neq 0$. For if $[c, d] \in D_M'$, then for every $[c', d'] \in D'$, $[c, d] \cdot [c', d'] = [c', d'] \cdot [c, d]$ implies $cc'd'd = cc'dd'$, and $d'd = dd'$ for every $d' \in D$. Thus $d \in D_M$. The converse is clear. Finally, the nonzero elements of D_M' form a multiplicative group since $[c_2, c_1] \cdot [c_1, c_2] = [c_1 c_2, c_1 c_2]$ the multiplicative identity of D'.

The proof of the theorem follows from the observation that $\mathfrak{D}'$ is in fact a near-field. For, let $\alpha=[c, d]$, $d\neq 0$, $\alpha_l=[d_l d, cd_l]$, $\alpha_r=[dd_r, cd_r]$. Then α_l and α_r are the left and right multiplicative inverses of α and $\alpha_r=(\alpha_l\alpha)\alpha_r=\alpha_l(\alpha\alpha_r)=\alpha_l$.

It is easily checked that the right distributive law holds in $\mathfrak{D}'$ if it holds in $\mathfrak{D}$. Hence a *D-ring which is a ring is isomorphic to a sub-(near)-ring of a division ring.*

7. Ideals and homomorphisms. In this section the concept of near-ring homomorphism is considered. The theory is analogous to the homomorphism theory for rings. However, the kernel of a homomorphism need not be a two-sided ideal in the usual sense. Neither need a two-sided ideal be the kernel of a near-ring homomorphism. Ideals will be defined (as in ring theory) so that the kernel of a homomorphism is an ideal and every ideal is the kernel of a homomorphism.

The *left coset* $a+S$, $a\in P$, modulo a set $S\subset P$ is the set $\{a+s \mid s\in S\}$. The *right coset* $S+a$ is defined analogously. A set S, or a near-ring $\mathfrak{S}$, is (*left*) *right invariant* in $\mathfrak{P}$ if $(Sa\subset S)aS\subset S$, $a\in P$. A sub-near-ring $\mathfrak{N}$ of $\mathfrak{P}$ is normal if $(N, +)$ is a normal sub-group of $(P, +)$. A (*left*) *right ideal* $\mathfrak{I}=(I, +, \cdot)$ of $\mathfrak{P}$ is a (left) right invariant normal sub-near-ring of $\mathfrak{P}$. A *two-sided ideal* is a right ideal which is also a left ideal. A *pseudo-right ideal* $\mathfrak{J}$ is a normal sub-near-ring of $\mathfrak{P}$ such that $ia-0a\in I$ for every $a\in P$, $i\in I$. A set S is *invariant* in $\mathfrak{P}$ if $(a+s_1)(b+s_2)-ab\in S$ for every a, $b\in P$ and s_1, $s_2\in S$. An *ideal* in $\mathfrak{P}$ is an invariant normal sub-near-ring of $\mathfrak{P}$. It should be noted that the normality of an ideal implies that the condition $(s_1+a)(s_2+B)-ab\in S$, a, $b\in P$, s_1, $s_2\in S$, is equivalent to the condition for invariance.

(1) *Every ideal* (*invariant sub-near-ring*) *is a left ideal* (*left invariant sub-near-ring*) *and pseudo-right ideal* (*pseudo-right invariant sub-near-ring*). (ii) *Every right ideal* (*right invariant sub-near-ring*) *is a pseudo-right ideal* (*pseudo-right invariant sub-near-ring*). The proof of (i) follows by taking special cases of the condition for invariance. Take $s_1=0$, $b=0$ to show that an ideal is left ideal and $a=0$, $s_2=0$ in the other case. Statement (ii) is trivial. Examples are now presented to show that, except for the relationships of this statement, the concepts of right invariance, left invariance, invariance and normality are independent. The near-ring $\mathfrak{Q}$ is a sub-near-ring of the near-ring $\mathfrak{P}$ having the properties stated and no others. The examples are easily verified.

(i) *Right invariant.* Let $\mathfrak{H}$ be a noncommutative nonnormal subgroup of a group $\mathfrak{G}$. Let $\mathfrak{P}$ be the sub-near-ring of $\mathfrak{T}_C(\mathfrak{G})$ such that $P=\{\alpha\in T_C(G) \mid g\alpha\in H\}$. Let $\mathfrak{Q}$ be the sub-near-ring of $\mathfrak{P}$ such that $Q=\{\beta_h\in P \mid h'\beta_h=h;\ h, h'\in H,\ h'\neq 0;\ 0\beta_h=0,\ g\beta_h=0,\ g\in H\}$.

(ii) *Left invariant.* Let $\mathfrak{H}$ be a nonnormal subgroup of $\mathfrak{G}$. Let $\mathfrak{P}=\mathfrak{T}_Z(\mathfrak{G})$. Let $\mathfrak{Q}$ be the sub-near-ring of $\mathfrak{P}$ such that $Q=\{\xi_h\in T_Z(G) \mid h\in H\}$. (*i.e.* $g\xi_h=h$; $h\in H$, $g\in G$.)

(iii) *Normal.* Let $\mathfrak{H}$ be a proper nontrivial subgroup of $\mathfrak{G}$. Let $\mathfrak{P}=\mathfrak{T}(\mathfrak{G})$ and $\mathfrak{Q}$ the sub-near-ring of $\mathfrak{P}$ such that $Q=\{\alpha\in T(G) \mid h\alpha=0,\ h\in H\}$. Another example is any subring of a ring which is neither a left or right ideal.

(iv) *Left invariant, invariant.* Let $\mathfrak{P}=(R\times S, \oplus, \odot)$ where $\mathfrak{R}=(R, +, \cdot)$ is a Z-ring and $\mathfrak{S}=(S, +, \cdot)$ is a near-ring with the multiplication $ab=0$, containing a nonnormal sub-near-ring $\mathfrak{T}$, and let $\oplus$, $\odot$ be defined by $(r_1, s_1)\oplus(r_2, s_2)=(r_1+r_2, s_1+s_2)$; $(r_1, s_1)\odot(r_2, s_2)=(r_2, 0)$. Let $\mathfrak{Q}$ be the sub-near-ring of $\mathfrak{P}$ such that $Q=\{(0, t)\,|\,t\in T\}$.

(v) *Right invariant, normal.* Let $\mathfrak{H}$ be a proper nontrivial subgroup of $\mathfrak{G}$. Let $\mathfrak{P}=\mathfrak{T}_C(\mathfrak{G})$ and let $\mathfrak{Q}$ be the sub-near-ring of $\mathfrak{P}$ such that $Q=\{\alpha\in T_C(G)\,|\,h\alpha=0, h\in H\}$. Another example is a right ideal (which is not an ideal) in a ring.

(vi) *Left invariant, normal.* Let $\mathfrak{H}$ be a proper, nontrivial subgroup of a commutative group $\mathfrak{G}$. Let $\mathfrak{P}=\mathfrak{T}(\mathfrak{G})$ and let $\mathfrak{Q}$ be the sub-near-ring of $\mathfrak{P}$ such that $Q=\{\xi_h\in T(G)\,|\,g\xi_h=h, g\in G, h\in H\}$. Another example is a left ideal (which is not a right ideal) in a ring.

(vii) *Right invariant, left invariant.* Let $\mathfrak{P}=\mathfrak{T}(\mathfrak{G})$, $\mathfrak{G}$ be a noncommutative group, and let $\mathfrak{Q}=\mathfrak{T}_Z(\mathfrak{G})$.

(viii) *Left invariant, invariant, normal.* Let $\mathfrak{H}$ be a normal subgroup of $\mathfrak{G}$. Let $\mathfrak{P}=\mathfrak{T}_Z(\mathfrak{G})$ and let $\mathfrak{Q}$ be the sub-near-ring of $\mathfrak{P}$ such that $Q=\{\xi_h\in P\,|\,g\xi_h=h, g\in G, h\in H\}$.

(ix) *Left invariant, right invariant, invariant.* Let $\mathfrak{P}$ be a near-ring with the multiplication $ab=0$, whose additive group is noncommutative and contains a normal subgroup $(Q, +)$. Let $\mathfrak{Q}$ be the sub-near-ring of $\mathfrak{P}$ determined by Q.

(x) *Right invariant, left invariant, normal.* Let $\mathfrak{G}$ be a commutative group. Let $\mathfrak{P}=\mathfrak{T}(\mathfrak{G})$ and $\mathfrak{Q}=\mathfrak{T}_Z(\mathfrak{G})$.

(xi) *Left invariant, right invariant, invariant, normal.* Let $\mathfrak{P}$ be a near-ring with the multiplication $ab=0$ or an R-ring, and let $\mathfrak{Q}$ be an ideal in $\mathfrak{P}$.

It was shown in (iv) that an invariant sub-near-ring need not be an ideal. However

THEOREM. *If $\mathfrak{P}$ is a near-ring with identity, then any invariant sub-near-ring is an ideal.*

The proof follows from the observation that $(a+i)(1+0)-a=a+i-a\in I$ for $a\in P$, $i\in I$. Relative to the definitions given, the homomorphism and ideal theory for rings carries over completely to near-rings. The very basic theorems of ring theory are now restated for near-rings.

THEOREM. *The cosets of $\mathfrak{P}$ modulo an ideal $\mathfrak{I}$ form a near-ring $\mathfrak{P}-\mathfrak{I}$ relative to the compositions* $(a+I)+(b+I)=(a+b+I)$, $(a+I)\cdot(b+I)=(ab+I)$.

The proof is similar to the corresponding proof for rings. Normality of $(I, +)$ replaces commutativity of addition and the invariance of $\mathfrak{I}$ is substituted for two-sided invariance. The near-ring $\mathfrak{P}-\mathfrak{I}$ is called the *difference near-ring* of $\mathfrak{P}$ modulo $\mathfrak{I}$.

A *homomorphism* of a near-ring $\mathfrak{P}$ into a near-ring $\mathfrak{Q}$ is a mapping of P into Q such that $(a+b)\theta=a\theta+b\theta$, and $(ab)\theta=(a\theta)\cdot(b\theta)$. An *isomorphism* (onto) is an (onto) 1-1 homomorphism. The near-ring $\mathfrak{P}$ is *isomorphic* to $\mathfrak{Q}(\mathfrak{P}\cong\mathfrak{Q})$ if there

exists an isomorphism from $\mathfrak{P}$ onto $\mathfrak{Q}$. The *kernel* of a homomorphism of $\mathfrak{P}$ to $\mathfrak{Q}$ is the inverse image of the identity of $(Q, +)$. The following theorems are examples of theorems which are true for rings and which carry over to near-rings. The proofs, which are essentially the same as in ring theory, are omitted.

If $\mathfrak{J}$ is an ideal in $\mathfrak{P}$, the natural map ν: $a\nu = a + I$, $a \in P$ is a homomorphism of $\mathfrak{P}$ onto $\mathfrak{P} - \mathfrak{J}$. The kernel of ν is $\mathfrak{J}$. Thus

THEOREM. *Every ideal is the kernel of a homomorphism (natural homomorphism). Conversely, the kernel of a homomorphism is an ideal.*

Let θ be a homomorphism of the near-ring $\mathfrak{P}$ into the near-ring $\mathfrak{Q}$ with kernel $\mathfrak{K}$ and image $\mathfrak{P}'$, and let $\mathfrak{J}$ be an ideal of $\mathfrak{P}$ contained in $\mathfrak{K}$. Then the map ϕ: $(a+I)\phi = a\theta$, $a \in P$, of $\mathfrak{P} - \mathfrak{J}$ onto $\mathfrak{P}'$ is a homomorphism. Further $\theta = \nu\phi$, and ϕ is an isomorphism if and only if $I = K$. Thus

THEOREM. *Every homomorphic image of a near-ring is isomorphic to a difference near-ring.*

8. Ideals in special classes of near-rings. A few theorems are now presented concerning ideals in special classes of near-rings. In the case of Z-rings the lattice structure of the ideals is completely determined.

(a) *Ideals in C-rings. A near-ring $\mathfrak{P}$ is a C-ring if and only if every ideal in $\mathfrak{P}$ is two-sided.* From Section 7, every ideal in $\mathfrak{P}$ is a left ideal and a pseudo-right ideal. In case $\mathfrak{P}$ is a C-ring a pseudo-right ideal is clearly a right ideal. Conversely, if every ideal in $\mathfrak{P}$ is two-sided, the ideal containing only the element 0 is two-sided. But then $0a = 0$ for every $a \in P$. It follows that *a near-ring is a C-ring if and only if the ideal $\{0\}$ is two-sided.*

(b) *Ideals in R-rings. Every ideal in a R-ring is two-sided and conversely every two-sided ideal in a R-ring is an ideal.* Since an R-ring is a C-ring, and every ideal in a C-ring is two-sided by the previous remarks, the first statement follows. To prove the converse, consider a two-sided ideal $\mathfrak{J}$ in $\mathfrak{P}$. Let p_1, $p \in P$, $i \in I$, and let p be either right distributive or anti-right distributive. Then the normality of $\mathfrak{J}$ insures that $(p_1 + i)p = p_1 p + i'$, $i' \in I$. Now consider any element $p_2 \in P$ and write $p_2 = p^1 + p^2 \cdots + p^n$ where p^i $(i = 1, \cdots, n)$ is right distributive or anti-right distributive. Then by the above remarks, and using the normality and right invariance of $I(p_1 + i_1)(p_2 + i_2) - p_1 p_2 = (p_1 + i_1)p_2 + (p_1 + i_1)i_2 - p_1 p_2 = (p_1 + i_1)(p^1 + \cdots + p^n) + i_3 - p_1 p_2 = (p_1 + i_1)p^1 + (p_1 + i_1)p^2 + \cdots + (p_1 + i_1)p^n + i_3 - p_1 p_2 = (p_1 p^1 + i_1') + (p_1 p^2 + i_2') + \cdots + (p_1 p^n + i_n') + i_3 - p_1 p_2 = p_1(p^1 + \cdots + p^n) + i_4 - p_1 p_2 = i_5$ where $i_1, i_2, \cdots, i_5, i_1', i_2', \cdots, i_n'$ are in I. This proves that $\mathfrak{J}$ is an ideal.

(c) *Ideals in Z-rings. Let $\mathfrak{P}$ be a Z-ring and $(Q, +)$ a normal subgroup of $(P, +)$. Then $\mathfrak{Q} = (Q, +, \cdot)$ is an ideal of $\mathfrak{P}$. Conversely every ideal of $\mathfrak{P}$ is characterized by a normal subgroup of $(P, +)$.* The system $\mathfrak{Q}$ is a sub-near-ring of $\mathfrak{P}$ since $\mathfrak{Q}$ is closed under multiplication. The normality assures that $\mathfrak{Q}$ is an ideal, for $(p_1 + q_1)(p_2 + q_2) - p_1 p_2 = p_2 + q_2 - p_2$ is in Q for $p_1, p_2 \in P$, $q_1, q_2 \in Q$. Conversely, if $\mathfrak{J}$ is an ideal, the group $(Q, +)$ is normal in $(P, +)$.

(d) *Annihilators and ideals.* Ideals in certain special near-rings may be constructed by various devices. For example, let $\mathfrak{P}$ be a near-ring, S a subset of P and $R(S)=\{a\in P \mid sa=0,\ s\in S\}$. (i) The system $\mathfrak{R}(S)=(R(S),\ +,\ \cdot)$ is a sub-near-ring of $\mathfrak{P}$ if $0\in S$ or if $\mathfrak{P}$ is a C-ring. (ii) If $\mathfrak{P}$ is a C-ring, then $\mathfrak{R}(S)$ is a right ideal. (iii) If S is right invariant and $\mathfrak{R}(S)$ a sub-near-ring of $\mathfrak{P}$, then $\mathfrak{R}(S)$ is an ideal. If $a,\ b\in R(S)$, then $a+b$ and $-a\in R(S)$, and if either $0\in S$ or $\mathfrak{P}$ is a C-ring then $ab\in R(S)$. This is easily checked. Statement (ii) follows from the observation that $s(ap)=(sa)p=0p=0$, and $s(-p+a+p)=-sp+0+sp=0$, for $p\in P,\ s\in S,\ a\in R(S)$. The proof of (iii) follows in a similar manner from a simple calculation.

The ideals constructed with annihilators can be generalized. Let $\mathfrak{P}$ be a near-ring and $S,\ M$ subsets of P. Let $R(S,\ M)=\{a\in P \mid sa\in M,\ s\in S\}$. (i) If $(M,\ +)$ is a sub-group of $(P,\ +)$ such that $M\subset S$ or M is right invariant, then the system $\mathfrak{R}(S,\ \mathfrak{M})=(R(S,\ M),\ +,\ \cdot)$ is a sub-near-ring of $\mathfrak{P}$. (ii) Let $M\subset S$ and $(M,\ +)$ be a subgroup of $(P,\ +)$. Then (a) if $(M,\ +)$ is normal in $(P,\ +)$, $\mathfrak{R}(S,\ M)$ is normal in $\mathfrak{P}$, (b) if M is right invariant, $\mathfrak{R}(S,\ M)$ is right invariant, (c) if S is right invariant, $\mathfrak{R}(S,\ M)$ is left invariant, (d) if S is right invariant and $(M,\ +)$ is normal in $(P,\ +)$, $\mathfrak{R}(S,\ M)$ is a left ideal, and (e) if $(M,\ +)$ is right invariant and normal in $(P,\ +)$, $\mathfrak{R}(S,\ M)$ is a right ideal. (iii) Let $\mathfrak{P}$ be an R-ring, $(S,\ +)$ a subgroup of $(P,\ +)$ right invariant in P, and $(M,\ +)$, $M\subset S$ a normal subgroup of $(P,\ +)$ right invariant in $\mathfrak{P}$. Then $\mathfrak{R}(S,\ M)$ is an ideal. In (i) it is easy to check that $(R(S,\ M),\ +)$ is a group if $(M,\ +)$ is a group. The condition that $M\subset S$ or is right invariant assures that the multiplication is closed. Thus (i) is proved. Suppose that $(M,\ +)$ is normal in $(P,\ +)$. Then $sm\in M$ and $s(p+a-p)=sp+sa-sp\in M$ for $s\in S,\ p\in P,\ a\in R(S,\ M)$. This implies that $(R(S,\ M),\ +)$ is normal. The right invariance of M implies that $s(ap)=(sa)p\in M$, while the right invariance of S implies that $s(pa)=(sp)a\in M$ for $s\in S,\ p\in P,\ a\in R(S,\ M)$. Statements (d) and (e) follow. To prove (iii), let $s\in S,\ p_1,\ p_2\in P,\ a_1,\ a_2\in R(S,\ M)$. Let $p_2=p^1+\cdots+p^n$ where p^i $(i=1,\ \cdots,\ n)$ is right or anti-right distributive. Then $sa_1=m_1\in M,\ sp_1+sa_1=s_2\in S,\ s_2a_2=m_2\in M$, so that, using the normality of $(M,\ +)$ and the properties of the p^i: $s((p_1+a_1)(p_2+a_2)-p_1p_2)=(sp_1+sa_1)p_2+(sp_1+sa_1)a_2-sp_1p_2=(sp_1+m_1)(p^1+\cdots+p^n)+s_2a_2-sp_1p_2=(sp_1+m_1)p^1+(sp_2+m_1)p^2+\cdots+(sp_1+m_1)p^n+m_2-sp_1p_2=sp_1(p^1+\cdots+p^n)+m_3-sp_1p_2=m_4$, where $m_3,\ m_4\in M$. The normality of $\mathfrak{R}(S,\ M)$ follows from (iia). It should be noted that the condition that $\mathfrak{P}$ be an R-ring can be weakened to the condition that P be weakly right distributive over S.

9. Problems and extensions. This section presents a few problems involving near-rings.

Blackett [2], and Deskins [5], have considered the problem of semi-simplicity and the radical for certain classes of near-rings. An acceptable definition for the radical of a general near-ring is still needed. The characterization of ideals, left ideals, *etc.* in special classes of near-rings also presents a problem. For exam-

ple, what is the class of near-rings characterized by the property that every two-sided ideal is an ideal, and conversely? Rings, R-rings and simple C-rings lie in this class.

It has been shown [1] that the transformation near-rings $\mathfrak{T}(\mathfrak{G})$ and $\mathfrak{T}_C(\mathfrak{G})$ are simple, whereas $\mathfrak{T}_R(\mathfrak{G})$ is not always simple. If $\mathfrak{G}$ is simple, is $\mathfrak{T}_R(\mathfrak{G})$, and in general, what characterizes the class of groups $\mathfrak{G}$ such that $\mathfrak{T}_R(\mathfrak{G})$ is simple? Similar questions can be asked about transformation near-rings generated by other special transformations, *e.g.* inner automorphisms, automorphisms.

Every group can be taken as the additive group of a near-ring. What characterizes the class of near-rings with the property that, if the group operation is taken as the addition of a near-ring, then the only possible multiplications are $ab=0$ or b? More generally, given a class of near-rings (*e.g.*, rings, C-rings), what is a characterization of the corresponding additive groups? This is certainly not a simple problem in the case of rings. Similar problems can be formulated for the multiplicative semi-group of a near-ring.

A class of "near-vector" spaces has already had application [7]. The general theory of "near-vector" spaces could be developed to generalize vector spaces. A further generalization might be to "near-modules." The analogy to Lie and Jordan rings could also be developed for near-rings. Similarly, near-field embeddings and Galois theory are problems to be considered. The fact that any class of operators on a group generate a near-ring leads one to surmise that near-rings might have application to nonlinear operators. Banach and topological algebras might be generalized to topological near-rings.

Veblen and Wedderburn used near-fields to construct the first example of a non-Desarguesian finite plane. Further studies of non-Desarguesian planes involving near-fields have recently been made [7, 9]. In any Veblen-Wedderburn geometry based on a near-field the little theorem of Desargues holds for all pairs of triangles in perspective from the point (1, 0, 0). Is there a 1-1 correspondence between near-fields and geometries with this property [9]? For other geometric problems related to near-fields see [7].

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A TOPOLOGY FOR SEQUENCES OF INTEGERS I

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1. Introduction. In what follows we shall be concerned with the set S of all sequences which possess a natural density. The natural density of the sequence $A=\{a_n\}$ is defined by $\delta(A)=\lim_{n\to\infty} n^{-1}A(n)$ whenever the limit exists and where $A(n)$ is the number of elements of A not exceeding n. It is also assumed that A is a strictly-increasing sequence of positive integers and that all sequences are infinite.

In the following sections we shall define a metric in the set S which depends on the concept of density. We then deduce a number of the simpler topological properties of the space S and attempt to determine its structure.

2. A metric space. If $A=\{a_n\}$ and $B=\{b_n\}$, the distance between A and B will be defined by $\rho(A, B)=0$ if $A=B$, *i.e.*, if $a_n=b_n$ for all n and by $\rho(A, B)=k^{-1}+|\delta(A)-\delta(B)|$ otherwise, where k is the smallest value of n such that $a_n\neq b_n$. Any real-valued function having certain simple properties could be used in place of $\delta(A)$ in this definition and many of the subsequent results would remain valid or require only slight modification. In particular, the replacement of the natural density by either the lower or upper asymptotic density would yield an extension of this metric to the set of all sequences.

The metric $\rho(A, B)=k^{-1}$ is well known and has many interesting properties. A discussion of this metric and references to the literature can be found in the books of Ostmann [1]. These books also contain a vast amount of other material on sequences and related topics as well as a very comprehensive bibliography. Excellent summaries of some of the arithmetical properties of sequences have been given by Erdös and Niven [2] and by Niven [3].

THEOREM 1. *S is a metric space.*

Proof. It is obvious that $\rho(A, B)\geqq 0$ and that $\rho(A, B)=0$ if and only if $A=B$. It is also clear that $\rho(A, B)=\rho(B, A)$. Let $\rho(A, C)=k_1^{-1}+|\delta(A)-\delta(C)|$, $\rho(A, B)=k_2^{-1}+|\delta(A)-\delta(B)|$, $\rho(B, C)=k_3^{-1}+|\delta(B)-\delta(C)|$. If one of $\rho(A, B)$ and

$\rho(B, C)$ is zero, then $\rho(A, C)$ is equal to the other. Also, $k_1 \geqq \min\{k_2, k_3\}$ and it follows that $k_1^{-1} \leqq k_2^{-1} + k_3^{-1}$. The triangle law follows since

$$|\delta(A) - \delta(C)| \leqq |\delta(A) - \delta(B)| + |\delta(B) - \delta(C)|.$$

COROLLARY. *The metric of S is not equivalent to the one given by $\rho(A, B) = 0$ if $A = B$ and $\rho(A, B) = k^{-1}$ otherwise, where k has the same meaning as before.*

COROLLARY. *S is a bounded metric space.*

THEOREM 2. *S is not totally bounded.*

Proof. Suppose that $S = \bigcup_{k=1}^{n} E_k$, where each E_k has diameter less than one. Then all sequences in any given E_k must have the same first term, say n_k. If A is any sequence such that $a_1 \neq n_k$ for $1 \leqq k \leqq n$, then $A \notin S$ and this is a contradiction.

COROLLARY. *S is not compact.*

We prove directly that S is not compact as follows. Let $E_n = \{A \in S \mid a_1 \geqq n\}$. It is obvious that $\{E_n\}$ is a collection of closed sets having the finite intersection property. But $\bigcap_{n=1}^{\infty} E_n$ is empty and so S is not compact. This argument actually shows that S is not countably compact.

The following definition is used in the formulation of the next theorem: $d(A) = \text{g.l.b. } n^{-1}A(n)$. We shall also make repeated use of the obvious fact that $d(A) \leqq \delta(A)$.

THEOREM 3. *Let*

$$A_n \to A. \tag{1}$$

Then

$$\limsup_{n \to \infty} d(A_n) \leqq d(A). \tag{2}$$

If the double limit

$$\alpha = \lim_{n,k \to \infty} k^{-1}A_n(k) \tag{3}$$

exists, then (1) *implies*

$$d(A_n) \to d(A). \tag{4}$$

Proof. Suppose that $\rho(A_n, A) = k_n^{-1} + |\delta(A_n) - \delta(A)|$. For every $\epsilon > 0$ there is a k_0 such that $k_0^{-1}A(k_0) < d(A) + \epsilon$. Since by (1), $k_n \to \infty$, there is an N_0 such that $k_n > k_0$ for every $n > N_0$. Thus

$$d(A_n) \leqq k_0^{-1}A_n(k_0) = k_0^{-1}A(k_0) < d(A) + \epsilon$$

if $n > N_0$. This yields (2).

We now assume (3) and first prove

$$\alpha = \delta(A). \tag{5}$$

Now $\lim_{k\to\infty} k^{-1}A_n(k)=\delta(A_n)$ for all n and it follows that $\lim_{n,k\to\infty} k^{-1}A_n(k)=\delta(A)$ since the double limit exists and $\delta(A_n)\to\delta(A)$ [5]. Starting from (3) and (5) we now prove (4). Let $\epsilon>0$. By (3) and (5) there is an N_1 and a K such that $k^{-1}A_n(k)>\delta(A)-\epsilon$ for every $n>N_1$, $k>K$. Since $k_n\to\infty$ by (1), we have $k_n>K$ for every $n>N_2\geqq N_1$. Hence

$$k^{-1}A_n(k) = k^{-1}A(k) \geqq d(A)$$

for all $n>N_2$ and all $k\leqq K$ and

$$k^{-1}A_n(k) > \delta(A) - \epsilon \geqq d(A) - \epsilon$$

for all $n>N_2$ and all $k>K$. Thus $k^{-1}A_n(k)>d(A)-\epsilon$ for all $n>N_2$ and all k. Therefore $d(A_n)\geqq d(A)-\epsilon$ for all $n>N_2$. Combining this with (2) we obtain (4).

It is easy to construct examples to show that condition (1) is not sufficient to yield (4) in the preceding theorem. As an application of the first part of the theorem, consider the set $F=\{A\in S \mid d(A)=\delta(A)\}$. Let $\{A_n\}$ be a sequence of elements in F and suppose that $A_n\to A$. Then $\delta(A)=\lim \delta(A_n)=\lim d(A_n)\leqq d(A)$. Hence $A\in F$ and F is a closed subset of S.

3. Topological properties of S. The proofs of several of the following results depend on the simple fact that there exists a sequence A such that $\delta(A)=\alpha$ for $0\leqq\alpha\leqq 1$.

THEOREM 4. *S is separable.*

Proof. For every rational number r choose a fixed sequence $\{a_{r,n}\}$ with the natural density r, $0\leqq r\leqq 1$. Let $S_{r,p}$ denote the class of sequences of the form $\{a_1, \cdots, a_t, a_{r,p+1}, a_{r,p+2}, \cdots\}$. Obviously every sequence in $S_{r,p}$ has the natural density r. Each class $S_{r,p}$ consists of a finite number of sequences. The number of sequences contained in each $\bigcup_p S_{r,p}$ is countable. As the set of rational numbers is countable, $\bigcup_{r,p} S_{r,p}$ is a countable union of countable sets and hence also countable. It remains to show that $\bigcup_{r,p} S_{r,p}$ is everywhere dense in S. Let $A=\{a_n\}\in S$. To every $\epsilon>0$ there corresponds an r such that $|\delta(A)-r|<\epsilon/2$. Choose $k>2/\epsilon$. If p is large enough, $S_{r,p}$ will contain the sequence $B=\{a_1, \cdots, a_k, a_{r,p+1}, a_{r,p+2}, \cdots\}$. Then

$$\rho(A, B) \leqq k^{-1} + |\delta(A) - \delta(B)| < (\epsilon/2) + (\epsilon/2) = \epsilon.$$

It can easily be shown that the set of all sequences which differ from a finite union of mutually disjoint arithmetic progressions in a finite number of terms is a countable everywhere-dense subset of S.

THEOREM 5. *S is nowhere locally compact.*

Proof. Let $A=\{a_n\}\in S$ and $N=\{B\in S \mid \rho(A, B)<\epsilon\}$, where $0<\epsilon<1$. Choose $B=\{b_n\}\in S$ such that $|\delta(A)-\delta(B)|=\epsilon/2$ and if n_k is the smallest value of n such that $a_k<b_n$ put $A_k=\{a_1, \cdots, a_k, b_{n_k}, b_{n_k+1}, \cdots\}$. Thus $\delta(A_k)=\delta(B)$ and $\{A_k\}$ is a Cauchy sequence. Since

$$\rho(A_k, A) \leqq k^{-1} + |\delta(A_k) - \delta(A)| = k^{-1} + \epsilon/2,$$

we have $A_k \in N$ for every $k > 2/\epsilon$. Since $\rho(A_k, A) > |\delta(A_k) - \delta(A)| = \epsilon/2$, A is not a limit point of $\{A_k\}$. As $\{A_k\}$ cannot have limit points different from A, $\{A_k\}$ has no limit points. Now if E is any open set containing A, then $N \subset E \subset \overline{E}$ if ϵ is small enough. Hence $\overline{E}$ is not sequentially compact and therefore not compact.

COROLLARY. *S is neither complete nor sequentially compact.*

THEOREM 6. *S is totally disconnected.*

Proof. Let E be any subset of S containing at least two elements. Now let E_1 denote the set of all sequences in E which agree with $A^* \in E$ in the first k or more places and let $E_2 = E - E_1$. It is clear that $A^* \in E_1$ for all k and that E_2 is nonempty for k sufficiently large. Also, $\rho(A_1, A_2) \geqq k^{-1}$, where $A_1 \in E_1$, $A_2 \in E_2$, and it follows that $\rho(E_1, E_2) \geqq k^{-1}$. But this implies that $\rho(\overline{E}_1, \overline{E}_2) \geqq k^{-1}$. Thus $E = E_1 \cup E_2$, where E_1 and E_2 are nonempty and satisfy the Hausdorff-Lennes condition. Hence E is not connected and S is totally disconnected.

COROLLARY. *Given any positive integer n, there exist sets E_1, E_2 such that $S = E_1 \cup E_2$ and $\rho(E_1, E_2) = n^{-1}$.*

THEOREM 7. *If $S = E_1 \cup E_2$, then $\rho(E_1, E_2) = 0$ or n^{-1}, where n is a positive integer.*

Proof. Suppose that $S = E_1 \cup E_2$ and let $d = \rho(E_1, E_2) > 0$. Then, given $\epsilon_n > 0$, there exists $A_n^1 \in E_1$ and $A_n^2 \in E_2$ such that

$$\rho(A_n^1, A_n^2) = k_n^{-1} + |\delta(A_n^1) - \delta(A_n^2)| < d + \epsilon_n.$$

Now let $\lambda_n = \frac{1}{2}|\delta(A_n^1) - \delta(A_n^2)|$ and choose a sequence A_n which agrees with A_n^1 and A_n^2 in the first $k_n - 1$ places but is different from both in the k_nth place and such that $\delta(A_n) = \frac{1}{2}\{\delta(A_n^1) + \delta(A_n^2)\}$. Then $\rho(A_n, A_n^1) = \rho(A_n, A_n^2) = \rho(A_n^1, A_n^2) - \lambda_n < d + \epsilon_n - \lambda_n$. Now let $\epsilon_n \to 0$ and suppose that zero is not a limit point of $\{\lambda_n\}$. Then $\rho(A_n, A_n^1) = \rho(A_n, A_n^2) < d$ for a sufficiently large value of n and it follows that $A_n \notin E_1 \cup E_2$ for this n and this is a contradiction. Now suppose that zero is a limit point of $\{\lambda_n\}$. Then there exists a subsequence $\{\lambda_{n_i}\}$ such that $\lambda_{n_i} \to 0$ and it follows that $k_{n_i}^{-1} \to d$. But this can happen only if $d = 0$ or if all but a finite number of $k_{n_i}^{-1}$ are equal to d.

Another problem that arises in connection with the space S is that of determining its dimension. As a consequence of Theorem 4 most of the well-known results of dimension theory could be used for this purpose. Also, in view of Theorem 5, it does not follow from Theorem 6 that S is zero-dimensional although it may seem that this should be the correct result. It will follow from the results of the next section that there exists a continuous one-one mapping of S onto a zero-dimensional subset of the unit interval.

4. Structure of S. Let $a=0\cdot\alpha_1\alpha_2\cdots\alpha_n\cdots$, where $\alpha_n=1$ if $n\in A$ and $\alpha_n=0$ otherwise. Also, let R denote the set of all real numbers (in the binary scale) in the interval $(0, 1]$ for which

$$(*) \qquad \lim_{n\to\infty} n^{-1}\sum_{i=1}^{n}\alpha_i$$

exists. Let the topology in R be determined by the metric $\rho(a, b)=0$ if $a=b$ and

$$\rho(a, b) = k^{-1} + \lim_{n\to\infty} n^{-1}\Big|\sum_{i=1}^{n}(\alpha_i-\beta_i)\Big|$$

otherwise, where $k=\max\left\{\sum_{i=1}^{m}\alpha_i, \sum_{i=1}^{m}\beta_i\right\}$ and m is the smallest value of n such that $\alpha_n\neq\beta_n$. It is clear that $\delta(A)$ exists if and only if the limit $(*)$ exists and in this case the two are equal. Since all sequences are infinite, the mapping $\pi: A\to a$ establishes a one-one correspondence between S and R and it is easy to see that π is an isometry. It follows from the remark following Theorem 4 that the rational numbers are everywhere dense in R. It is also clear that π is a continuous mapping of S into the unit interval and that R is a zero-dimensional subset of the unit interval.

Since the limit $(*)$ is equal to $\frac{1}{2}$ if and only if a is simply normal in the binary scale, it follows that R includes all such numbers and that all sequences corresponding to such numbers have natural density $\frac{1}{2}$. As a consequence of the fact that almost all numbers are normal [6, 7], we can conclude that $mR=1$ and that every sequence has a natural density $\frac{1}{2}$ with probability 1. The following results characterize to a certain extent the structure of the open and closed sets of S.

THEOREM 8. *All open and closed sets of R are measurable subsets of the unit interval.*

Proof. Let $S_k(A_0)$ be the set of all $A\in S$ which agree with A_0 in the first $k-1$ or more places; let

$$S_k(A_0, \gamma) = \left\{A\in S \,\middle|\, |\delta(A)-\delta(A_0)| < \gamma - k^{-1}\right\}$$

and

$$S(A_0, \gamma) = \left\{A\in S \,\middle|\, \rho(A, A_0) < \gamma\right\}.$$

Then

$$S(A_0, \gamma) = \bigcup_{k>1/\gamma}\left\{S_k(A_0)\cap S_k(A_0, \gamma)\right\}.$$

Now $\pi\left\{S_k(A_0)\right\}$ is an interval and $\pi\left\{S_k(A_0, \gamma)\right\}$ has measure zero or one. It follows that all the open spheres in R are measurable and, since R is separable, so are all open sets. Also, all closed sets of R are measurable since they are complements of open sets with respect to R and $mR=1$.

COROLLARY. *A subset E of the unit interval is measurable if $E \cap R$ is an open or closed subset of R.*

The proof of the next theorem is obtained by introducing suitable refinements in the proof of the preceding one and is therefore omitted.

THEOREM 9. *If E is an open (closed) set in R, then there exists an open (closed) subset E^* of the unit interval such that $m(E \cup E^* - E \cap E^*) = 0$.*

5. Multiplication of sequences. Finally, we shall consider briefly a certain binary operation on sequences which we shall call multiplication. If $A = \{a_n\}$, $B = \{b_n\}$, then define the product $AB = \{b_{a_n}\}$. This definition was introduced by Niven [3], who also proved that $\delta(AB)$ exists and that $\delta(AB) = \delta(A)\delta(B)$ whenever $\delta(A)$ and $\delta(B)$ exist. This result makes it possible to show that S is a topological structure of a type which has been studied rather extensively by Wallace [4] and others.

THEOREM 10. *S is a mob.*

Proof. It is obvious that the set of all sequences is a semigroup (with an identity and right cancellation). But S is closed under multiplication (by Niven's theorem) and is therefore a semigroup also. Since all metric spaces are Hausdorff spaces, all that remains is to show that multiplication is continuous in the topology of S. Let $\rho(A_n, A) \to 0$, $\rho(B_n, B) \to 0$, where $A = \{a_i\}$, $A_n = \{a_{n,i}\}$, $B = \{b_i\}$ and $B_n = \{b_{n,i}\}$. Then

$$\rho(A_nB_n, AB) = k_n^{-1} + |\delta(A_nB_n) - \delta(AB)|$$

$$= k_n^{-1} + |\delta(A_n)\delta(B_n) - \delta(A)\delta(B)| \to 0,$$

where k_n is the smallest value of k such that $b_{n,a_{n,k}} \neq b_{a_k}$.

It has also been shown by Niven [3] that $d(AB) \geqq d(A)d(B)$. Suppose that $d(A) = \delta(A)$, $d(B) = \delta(B)$. Then $\delta(AB) = \delta(A)\delta(B) = d(A)d(B) \leqq d(AB)$ and consequently $d(AB) = \delta(AB)$, *i.e.*, the set $F = \{A \in S \mid d(A) = \delta(A)\}$ is closed under multiplication.

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ON THE NUMERICAL SOLUTION OF DIRICHLET-TYPE PROBLEMS

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1. Introduction. A study of the solution of

$$\frac{\partial^2 u}{\partial z^2}+\frac{\partial^2 u}{\partial \rho^2}-(K/\rho)\frac{\partial u}{\partial \rho}\equiv u_{zz}+u_{\rho\rho}-(K/\rho)u_\rho=0,\ \rho\neq 0 \tag{1.1}$$

for K a fixed, real, nonzero constant is of considerable interest. Applications occur in potential theory for $K=-1$, in fluid dynamics and theory of heat flow for $K=1$, and in certain problems of stress for $K=3,\ -3,\ -5$ (see [1]).

The problem to be considered is a Dirichlet-type problem. In the $z\rho$-plane, let G be a closed, bounded, simply-connected plane region whose interior is denoted by R and whose boundary curve is denoted by S. Let G not contain any point where $\rho=0$. Let $g(z,\ \rho)$ be defined and continuous on S. The problem, then, is to produce a function $u(z,\ \rho)$ such that for fixed K

(a) $$u(z,\ \rho)\equiv g(z,\ \rho) \text{ on } S$$

and

(b) $$u(z,\ \rho) \text{ satisfies equation (1.1) in } G.$$

Under quite general conditions, there exists a unique solution and only such cases will be considered [2], [3]. However, the analytical determination of $u(z,\ \rho)$ is quite another story from that of its existence and usually offers what are, at present, insurmountable problems. The approach here, then, will be from a numerical analysis point of view and will be general enough to apply for arbitrary, but fixed, nonzero K and will utilize the rectangular type grid.

2. The numerical method. Let h and d be fixed positive constants and let $(z_0,\ \rho_0)$ be an arbitrary, but fixed, point of G. Denote by G_h the set of all points of the form $(z_0+mh,\ \rho_0+nd)$ contained in G, where m and n are integers. Two points $(z_1,\ \rho_1)$ and $(z_2,\ \rho_2)$ of G_h are called adjacent if and only if the straight line segment joining them is contained in G and

(a) $$(z_1-z_2)^2+(\rho_1-\rho_2)^2=h^2 \text{ and } \rho_1=\rho_2$$

or

(b) $$(z_1-z_2)^2+(\rho_1-\rho_2)^2=d^2 \text{ and } z_1=z_2.$$

The interior of G_h, denoted by R_h, is the set of all points of G_h which have four adjacent points in G_h. The boundary of G_h, denoted by S_h, is the set defined by $G_h=R_h\cup S_h,\ R_h\cap S_h=\theta$.

Suppose G_h consists of n points. Number these in a one-to-one fashion with the integers $1, \cdots, n$. Denote the coordinates of the point numbered k by (z_k, ρ_k) and the unknown function u at (z_k, ρ_k) by $u(z_k, \rho_k) \equiv u_k$, for $k = 1, \cdots, n$.

Let (z_i, ρ_i) be an arbitrary point of S_h, the lattice boundary. Approximate u_i by $g(z', \rho')$, where (z', ρ') is the nearest point of S to (z_i, ρ_i). If (z', ρ') is not unique, choose any one of the set of nearest points and use it. The problem of finding numerical approximations to $u(z, \rho)$ on S_h is, though crudely done, adequate for present purposes.

We require then at each point (z_i, ρ_i) of R_h, that u satisfy

$$
\begin{aligned}
&-2(1 + p^{-2})u(z_i, \rho_i) + p^{-2}[u(z_i + h, \rho_i) + u(z_i - h, \rho_i)] \\
&+ \left(1 - \frac{dK}{2\rho}\right) u(z_i, \rho_i + d) + \left(1 + \frac{dK}{2\rho}\right) u(z_i, \rho_i - d) = 0
\end{aligned} \tag{2.1}
$$

where $h = pd$. Application of equation (2.1) to each point of R_h yields a system of linear algebraic equations which, when solved, yields the remaining numerical approximations of the analytical solution.

3. Derivation of difference analogue (2.1). We seek to approximate a solution $u(z, \rho)$ which is of class C^4 and use the notation

$$
i!j!A_{i,j} \equiv u_{i,j}; \qquad u_{i,j} \equiv \frac{\partial^{i+j} u}{\partial z^i \partial \rho^j}. \tag{3.1}
$$

Let (z, ρ) be an arbitrary point of R_h. The aim will be to develop a "5-point" analogue, so consider (z, ρ) and its four adjacent points. Let (z, ρ), $(z+h, \rho)$, $(z, \rho+d)$, $(z-h, \rho)$, $(z, \rho-d)$ be denoted, respectively, by 0, 1, 2, 3, 4. Hence, $(z_0, \rho_0) = (z, \rho)$, $(z_1, \rho_1) = (z+h, \rho)$, $(z_2, \rho_2) = (z, \rho+d)$, $(z_3, \rho_3) = (z-h, \rho)$, $(z_4, \rho_4) = (z, \rho-d)$. Also let

$$
h = pd, \; h > 0, \; d > 0, \; p > 0. \tag{3.2}
$$

To produce a difference equation which approximates (1.1), let

$$
L(u) \equiv \sum_0^4 \alpha_i u_i = 0. \tag{3.3}
$$

By use of equation (3.2), the definition of $A_{i,j}$ in (3.1), and the Taylor expansions of u_1, u_2, u_3, u_4 one may rewrite equation (3.3) as

$$
\begin{aligned}
L(u) \equiv\; & A_{0,0}(\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) + d[A_{1,0}p(\alpha_1 - \alpha_3) + A_{0,1}(\alpha_2 - \alpha_4)] \\
& + d^2[A_{2,0}p^2(\alpha_1 + \alpha_3) + A_{0,2}(\alpha_2 + \alpha_4)] \\
& + d^3[A_{3,0}p^3(\alpha_1 - \alpha_3) + A_{0,3}(\alpha_2 - \alpha_4)] + O(d^4) = 0.
\end{aligned} \tag{3.4}
$$

Equation (1.1) may be rewritten as: $A_{2,0} = KA_{0,1}/(2\rho) - A_{0,2}$. Substitution of this latter expression for $A_{2,0}$ in equation (3.4) yields:

$$
\begin{aligned}
L(u) \equiv\ & A_{0,0}(\alpha_0+\alpha_1+\alpha_2+\alpha_3+\alpha_4) \\
& + d\left\{A_{1,0}p(\alpha_1-\alpha_3)+A_{0,1}\left[\alpha_2-\alpha_4+(\alpha_1+\alpha_3)\frac{Kp^2d}{2\rho}\right]\right\} \\
& + d^2A_{0,2}[\alpha_2+\alpha_4-p^2(\alpha_1+\alpha_3)] \\
& + d^3[A_{3,0}p^3(\alpha_1-\alpha_3)+A_{0,3}(\alpha_2-\alpha_4)]+O(d^4)=0.
\end{aligned} \tag{3.5}
$$

Let

$$
\begin{aligned}
\alpha_0+\alpha_1+\alpha_2+\alpha_3+\alpha_4 &= \epsilon_1 = O(d^4), \\
\alpha_2-\alpha_4+(\alpha_1+\alpha_3)\frac{Kp^2d}{2\rho} &= \epsilon_2 = O(d^3), \\
\alpha_1-\alpha_3 &= \epsilon_3 = O(d^2), \\
\alpha_2+\alpha_4-p^2(\alpha_1+\alpha_3) &= \epsilon_4 = O(d^2), \\
\alpha_2-\alpha_4 &= \epsilon_5 = O(d).
\end{aligned}
$$

The solution of this linear system is

$$
\begin{aligned}
\alpha_1 &= \epsilon_3/2+\rho\epsilon_2/(Kp^2d)-\rho\epsilon_5/(Kp^2d), \\
\alpha_2 &= \epsilon_4/2+\epsilon_5/2+\rho\epsilon_2/(Kd)-\rho\epsilon_5/(Kd), \\
\alpha_3 &= -\epsilon_3/2+\rho\epsilon_2/(Kp^2d)-\rho\epsilon_5/(Kp^2d), \\
\alpha_4 &= \epsilon_4/2-\epsilon_5/2+\rho\epsilon_2/(Kd)-\rho\epsilon_5/(Kd), \\
\alpha_0 &= \epsilon_1-2\rho\epsilon_2/(Kp^2d)+2\rho\epsilon_5/(Kp^2d)-\epsilon_4-2\rho\epsilon_2/(Kd)+2\rho\epsilon_5/(Kd).
\end{aligned}
$$

Substitution of these values into equation (3.3) yields the general difference analogue

$$
\begin{aligned}
& [\epsilon_1-\epsilon_4-2\rho\epsilon_2/(Kp^2d)+2\rho\epsilon_5/(Kp^2d)-2\rho\epsilon_2/(Kd)+2\rho\epsilon_5/(Kd)]u_0 \\
& \quad +[\epsilon_3/2+\rho\epsilon_2/(Kp^2d)-\rho\epsilon_5/(Kp^2d)]u_1 \\
& \quad +[\epsilon_4/2+\epsilon_5/2+\rho\epsilon_2/(Kd)-\rho\epsilon_5/(Kd)]u_2 \\
& \quad +[-\epsilon_3/2+\rho\epsilon_2/(Kp^2d)-\rho\epsilon_5/(Kp^2d)]u_3 \\
& \quad +[\epsilon_4/2-\epsilon_5/2+\rho\epsilon_2/(Kd)-\rho\epsilon_5/(Kd)]u_4=0.
\end{aligned} \tag{3.6}
$$

In order to obtain equation (2.1) from equation (3.6), let $\epsilon_1=\epsilon_2=\epsilon_3=\epsilon_4=0$, $\epsilon_5=-dK$, divide through by ρ, and note that $u_0=u(z,\ \rho)$, $u_1=u(z+h,\ \rho)$, $u_2=u(z,\ \rho+d)$, $u_3=u(z-h,\ \rho)$, $u_4=u(z,\ \rho-d)$.

It is also now quite apparent, from equation (3.6), why the discussion here does not yield the well-known results for $K=0$ as a special case.

4. Theorems on the numerical method of Section 2. In this section let $\bar{\rho}=GLB_{\rho\in G}\,|\rho|$. By definition of G, $\bar{\rho}>0$. Let $u(z,\ \rho)$ be the solution of the Dirichlet type problem being considered, and let $U(z,\ \rho)$ be the solution of the numerical method described in Section 2. This latter assumption means that if

(z, ρ) is a point of R_h and $U_0 = U(z, \rho)$, $U_1 = U(z+h, \rho)$, $U_2 = U(z, \rho+d)$, $U_3 = U(z-h, \rho)$, $U_4 = U(z, \rho-d)$, then

$$(4.1) \quad -2(1+p^{-2})U_0 + p^{-2}(U_1+U_3) + \left(1-\frac{dK}{2\rho}\right)U_2 + \left(1+\frac{dK}{2\rho}\right)U_4 = 0.$$

THEOREM 1. *The solution of the system of linear equations which results by application of the method of Section 2 is unique if* $0 < d < 2\bar{\rho}/|K|$.

Proof. It is sufficient to show that the determinant of the system of linear equations is not zero and this is done by demonstrating that the only solution of the homogeneous system which results by considering $g(z, \rho) \equiv 0$ on S is the zero vector. Suppose then there exists a nontrivial solution for the homogeneous system. For some point of R_h, $U \neq 0$. Suppose $U > 0$. Let the largest value occur at some point, say, (z_0, ρ_0). Then

$$(4.2) \quad 2(1+p^{-2})U_0 = p^{-2}(U_1+U_3) + \left(1-\frac{dK}{2\rho}\right)U_2 + \left(1+\frac{dK}{2\rho}\right)$$

and

$$(4.3) \qquad M = U_0 \geqq U_i, \qquad i = 1, 2, 3, 4.$$

Case 1. Suppose $U_1 = U_2 = U_3 = U_4$. Then (4.2) becomes

$$(4.4) \qquad 2(1+p^{-2})U_0 = 2(1+p^{-2})U_1.$$

Hence $U_0 = U_1 = U_2 = U_3 = U_4 = M$.

Case 2. Suppose not all of U_1, U_2, U_3, U_4 are equal. Then one is a maximum If U_1 is maximum, let $U_2 = U_1 - k_2$, $U_3 = U_1 - k_3$, $U_4 = U_1 - k_4$, where k_2, k_3, k_4 are nonnegative and at least one is positive. Then (4.2) becomes

$$(4.5) \quad 2(1+p^{-2})U_0 = 2(1+p^{-2})U_1 - k_3p^{-2} - k_2\left(1-\frac{dK}{2\rho}\right) - k_4\left(1+\frac{dK}{2\rho}\right).$$

However, since $d < 2\bar{\rho}/|K|$, it follows that $(1-dK/2\rho)$ and $(1+dK/2\rho)$ are positive. Hence from (4.5) it follows that $2(1+p^{-2})U_0 < 2(1+p^{-2})U_1$ or, equivalently, that $U_0 < U_1$, which contradicts (4.3). Hence $U_0 = U_1 = U_2 = U_3 = U_4 = M$.

If U_2 or U_3 or U_4 is selected as the maximum, analogous reasoning yields the same result. So, Case 2 is not possible.

Continuing in an analogous fashion, one may show in a finite number of steps that U at a boundary point has the value M. But this contradicts the fact that U at every point of S_h is zero, by the method of approximation for points of S_h. Hence the theorem is proved.

Note that Theorem 1 establishes a practical, sufficient condition which assures a unique numerical solution. In all that follows, then, it is assumed that $d < 2\bar{\rho}/|K|$.

We proceed now by generalizing methods, ideas, and results of Gerschgorin [4].

DEFINITION. Let

$$L[u(z,\rho)] \equiv d^{-2}\left[-2(1+\rho^{-2})u(z,\rho)+\rho^{-2}u(z+h,\rho)+\rho^{-2}u(z-h,\rho)+\left(1-\frac{dK}{2\rho}\right)u(z,\rho+d)+\left(1+\frac{dK}{2\rho}\right)u(z,\rho-d)\right].$$

LEMMA 1. *If $L[v]\leqq 0$ on R_h and $v\geqq 0$ on S_h, then $v\geqq 0$ on R_h.*

The proof follows immediately as in [3] or [5].

LEMMA 2. *If $-|L[v_1]|\geqq L[v_2]$ in R_h and $|v_1|\leqq v_2$ on S_h, then $|v_1|\leqq v_2$ on R_h.*

The proof follows immediately as in [3] or [5].

LEMMA 3. *If $|L[v]|\leqq A$ on R_h, $|v|\leqq B$ on S_h and r is the radius of any circle of the kind described below, which contains G, then $|v|\leqq Ar^2/4+B$, on R_h.*

Proof. By assumption, G is closed, bounded, simply-connected, and contains no point of the form $(z, 0)$. Hence, it follows that for all (z, ρ) in G, ρ has the same sign and also has a finite maximum and a finite minimum in G. Select, now, a circle $(z-a)^2+(\rho-b)^2=r^2$ which contains G, such that b satisfies the inequality

$$1-\frac{K}{2\rho}(\rho-b)\geqq 1 \tag{4.6}$$

for all ρ in G. At least one such circle exists since G is bounded, since $\rho\neq 0$, and since b may be readily selected as follows:

(a) choose $b\geqq\rho^*$; if ρ^* and K are of the same sign, where $\rho^*=\max_{\rho\in G}\rho$,
(b) choose $b\leqq\bar{\rho}$; if $\bar{\rho}$ and K are of different sign, where, $\bar{\rho}=\min_{\rho\in G}\rho$.

Now let

$$w(z,\rho)=\left[\frac{Ar^2}{4}\left\{1-\frac{(z-a)^2+(\rho-b)^2}{r^2}\right\}+B\right].$$

Direct calculation yields $L[w(z,\rho)]=-A[1-(K/2\rho)(\rho-b)]$. However, since $A\geqq 0$, by assumption, and by (4.6), it follows that $L[w(z,\rho)]\leqq -A$. Since, then, $|L[v]|\leqq A$ on R_h, by assumption, it follows that, on R_h, $-|L[v]|\geqq L[w]$. Moreover, since $|v|\leqq B$ on S_h and $w\geqq B$ on S_h, $w\geqq|v|$. Hence $-|L[v]|\geqq L[w]$ on R_h and $w\geqq|v|$ on S_h. By Lemma 2, $w\geqq|v|$ on R_h, or $|v|\leqq w\leqq Ar^2/4+B$, on R_h, which proves the lemma.

THEOREM 2. *If $u(z,\rho)$ is of class C^4, $\rho\neq 0$ in G, u denotes the solution of the Dirichlet type problem described in Section* 1, *U denotes the solution of the linear*

system which results by application of the method of Section 2, and $d<2\bar{\rho}/|K|$, *then, on* R_h,

$$|U-u| \leqq (r^2/4)[M_4h^2/12 + M_4d^2/12 + M_3d^2|K|/(6\bar{\rho}) + M_4d^3|K|/(24\bar{\rho})] + M_1(h+d), \tag{4.7}$$

where

$$M_k = \max_{i,j}\left[\max_{(z,\rho)\in G}|u_{i,j}|\right], \qquad i+j=k;\ i, j \text{ nonnegative integers},$$

$\bar{\rho}=GLB_{\rho\in G}|\rho|$, *and* r *is the radius of any circle of type described in the proof of Lemma* 3.

Proof. Let $P=L[u]-(u_{zz}+u_{\rho\rho}-(K/\rho)u_\rho)$. Substitution of finite Taylor series expansions for $u(z+h,\rho)$, $u(z-h,\rho)$, $u(z,\rho+d)$, $u(z,\rho-d)$ in $L[u]$, yields

$$P = \frac{h^2}{4!}u_{4,0}(\beta_1,\rho) + \frac{h^2}{4!}u_{4,0}(\beta_2,\rho) + \frac{d^2}{4!}u_{0,4}(z,\gamma_1) + \frac{d^2}{4!}u_{0,4}(z,\gamma_2) - \frac{d^2Ku_{0,3}(z,\rho)}{6\rho} + \frac{Kd^3u_{0,4}(z,\gamma_2)}{(2\rho)4!} - \frac{Kd^3u_{0,4}(z,\gamma_1)}{(2\rho)4!}.$$

Hence $|P| \leqq M_4h^2/12+M_4d^2/12+M_3d^2|K|/(6\bar{\rho})+|K|M_4d^3/(4!\bar{\rho})$.

Now, since $u_{zz}+u_{\rho\rho}-(K/\rho)u_\rho=0$, then

$$|L[u]-(u_{zz}+u_{\rho\rho}-(K/\rho)u_\rho)| = |L[u]| = |P| \leqq M_4h^2/12 + M_4d^2/12 + M_3d^2|K|/(6\bar{\rho}) + M_4d^3|K|/(24\bar{\rho}). \tag{4.8}$$

Also, for any point of S_h, U was selected as the value of $g(z',\rho')$ at some nearest point (z',ρ') on S, and $g(z',\rho')=u(z',\rho')$ on S. Hence for any point (z,ρ) of S_h

$$|U(z,\rho)-u(z,\rho)| = |g(z',\rho')-u(z,\rho)| = |u(z',\rho')-u(z,\rho)| \tag{4.9}$$

where (z',ρ') is a point of S and

$$(z-z')^2+(\rho-\rho')^2 \leqq h^2 \quad \text{or} \quad (z-z')^2+(\rho-\rho')^2 \leqq d^2.$$

Therefore

$$\begin{aligned} |U(z,\rho)-u(z,\rho)| &= |u(z',\rho')-u(z,\rho')+u(z,\rho')-u(z,\rho)| \\ &\leqq |u_z(\delta_1,\rho')(z-z')| + |u_\rho(z,\delta_2)(\rho-\rho')| \\ &\leqq M_1h + M_1d = M_1(h+d). \end{aligned} \tag{4.10}$$

It must also be noted that $L[U]=0$, by equation (4.1). Hence

$$|L[u-U]| = |L[u]-L[U]| = |L[u]|. \tag{4.11}$$

Applying Lemma 3 to equations (4.8), (4.10) and (4.11), one finds that on R_h $|U-u| \leqq (r^2/4)[M_4h^2/12 + M_4d^2/12 + M_3d^2|K|/(6\bar{\rho}) + M_4d^3|K|/(24\bar{\rho})] + M_1(h+d)$, which proves the theorem.

THEOREM 3. *Under the conditions of Theorem 2, the numerical solution U converges to the analytical solution u as* $(h^2+d^2)\to 0$.

The proof follows directly from inequality (4.7) and from the method described for approximating $u(z, \rho)$ on S_h.

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MATHEMATICAL NOTES

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ON THE TURAN INEQUALITY FOR CERTAIN POLYNOMIALS

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1. Let us define for the sequence of functions $\{f_n(x)\}$ the expression

$$\Delta_n(f) = f_n^2(x) - f_{n+1}(x)f_{n-1}(x) \qquad (n \geqq 1).$$

Szegö [4] proved that $\Delta_n(P) \geqq 0$ $(-1 \leqq x \leqq 1;\ n \geqq 1)$, where $P_n(x)$ is the Legendre polynomial, and indicated that the same inequality holds for the Hermite, Laguerre, and ultraspherical polynomials. The latter implies that the derivatives of the Legendre polynomials also satisfy the Turan inequality [2].

We shall prove here that the derivatives of a certain class of polynomials satisfy the Turan inequality. This class includes the Hermite and the Tchebycheff polynomials as special cases. Only the knowledge of the recurrence relation shall be required. We shall also indicate that the derivatives of the Laguerre polynomials have this property.

2. Let us prove first the following:

LEMMA. *If* $\{p_n(x)\}$ *are the polynomials defined by*

$$p_{n+1}(x) = (A_n x + B_n)p_n(x) - C_n p_{n-1}(x) \qquad (n \geqq 1), \tag{2.1}$$

where $p_0(x)=1$, $p_1(x)=A_0x+B_0$, $C_n>0$ *and* $A_n>0$, *then*

$$\lambda_{n+1,k}(x) = p_{n+1}^{(k)}(x)p_n^{(k-1)}(x) - p_{n+1}^{(k-1)}(x)p_n^{(k)}(x) > 0$$
$$(k = 1, 2, 3, \cdots, n+1;\ n \geqq 1),$$

where $p_n^{(k)}(x)$ *indicates the kth derivative of* $p_n(x)$.

Proof. If we differentiate (2.1) k times, we can see easily that

$$\frac{p_{n+1}^{(k)}(x)p_n^{(k)}(y) - p_{n+1}^{(k)}(y)p_n^{(k)}(x)}{x-y}$$
$$= A_n p_n^{(k)}(x)p_n^{(k)}(y) + kA_n \frac{p_n^{(k-1)}(x)p_n^{(k)}(y) - p_n^{(k-1)}(y)p_n^{(k)}(x)}{x-y}$$
$$+ C_n \frac{p_n^{(k)}(x)p_{n-1}^{(k)}(y) - p_n^{(k)}(y)p_{n-1}^{(k)}(x)}{x-y}.$$

Taking the limit as $x \to y$, we get

$$\lambda_{n+1,k+1}(x) = A_n[p_n^{(k)}(x)]^2 + C_n\lambda_{n,k+1}(x) + kA_n\Gamma_{n,k}(x), \tag{2.2}$$

where $\Gamma_{n,k}(x) = [p_n^{(k)}(x)]^2 - p_n^{(k+1)}(x)p_n^{(k-1)}(x)$.

Now for any sequence of polynomials $\{f_n(x)\}$ whose zeros are all real and simple we have

$$[f_n'(x)]^2 - f_n(x)f_n''(x) > 0 \qquad (-\infty < x < \infty;\ n \geqq 1).$$

But since $p_n(x)$ has all its zeros real ([3], p. 44) then the zeros of $p_n^{(k-1)}(x)$ are all real and consequently

$$\Gamma_{n,k}(x) > 0 \qquad (-\infty < x < \infty;\ k = 1, \cdots, n;\ n \geqq 1). \tag{2.3}$$

We also note that $\lambda_{k-1,k}(x)=0$, and

$$\lambda_{k,k}(x) = p_k^{(k)}(x)p_{k-1}^{(k-1)}(x) = k!(k-1)!\prod_0^{k-1} A_r \prod_0^{k-2} A_s > 0. \tag{2.4}$$

Thus it follows by induction from (2.2), (2.3), and (2.4) that

$$\lambda_{n,k}(x) > 0 \qquad (-\infty < x < \infty;\ k = 1, \cdots, n,\ n \geqq 1).$$

This completes the proof of the lemma.

3. Now consider the polynomials defined by

$$p_{n+1}(x) = xA_np_n(x) - C_np_{n-1}(x) \qquad (n \geqq 1),$$
$$p_0(x) = 1, \qquad p_1(x) = A_0x, \qquad A_n \neq 0. \tag{3.1}$$

If we differentiate (3.1) k times, we get, leaving out simple calculations,

$$\begin{aligned}\Delta_n(p^{(k)}) &= C_{n-1}\Delta_{n-1}(p^{(k)}) + kA_{n-1}\lambda_{n,k}(x) + (C_n - C_{n-1})[p_{n-1}^{(k)}(x)]^2 \\ &\quad + (A_{n-1} - A_n)[kp_n^{(k-1)}(x) + xp_n^{(k)}(x)]p_{n-1}^{(k)}(x) \\ &= C_{n-1}\Delta_{n-1}(p^{(k)}) + (C_n - C_{n-1})[p_{n-1}^{(k)}(x)]^2 + kA_{n-1}\lambda_{n,k}(x) \\ &\quad + \left(\frac{A_{n-1}}{A_n} - 1\right)p_{n-1}^{(k)}(x)[p_{n+1}^{(k)}(x) + C_n p_{n-1}^{(k)}(x)] \\ &= C_{n-1}\Delta_{n-1}(p^{(k)}) + \left(\frac{A_{n-1}C_n - A_nC_{n-1}}{A_n}\right)[p_{n-1}^{(k)}(x)]^2 \\ &\quad + kA_{n-1}\lambda_{n,k}(x) + \left(1 - \frac{A_{n-1}}{A_n}\right)\Delta_n(p^{(k)}) + \left(\frac{A_{n-1}}{A_n} - 1\right)[p_n^{(k)}(x)]^2.\end{aligned}$$

Hence

$$\begin{aligned}\frac{A_{n-1}}{A_n}\Delta_n(p^{(k)}) &= C_{n-1}\Delta_{n-1}(p^{(k)}) + kA_{n-1}\lambda_{n,k}(x) \\ &\quad + \left(\frac{A_{n-1}C_n - A_nC_{n-1}}{A_n}\right)[p_{n-1}^{(k)}(x)]^2 + \left(\frac{A_{n-1}}{A_n} - 1\right)[p_n^{(k)}(x)]^2.\end{aligned} \tag{3.2}$$

THEOREM. *Let $p_n(x)$ be defined by* (3.1) *with the added conditions that $C_n>0$, $A_n>0$, $C_nA_{n-1}-C_{n-1}A_n\geqq 0$, $A_n\leqq A_{n-1}$. Then*

$$\Delta_n(p^{(k)}) \geqq \left\{k!\prod_{r=0}^{k-1}A_r\right\}^2\cdot\{C_kC_{k-1}\cdots C_{n-1}\}\{A_n/A_k\} \tag{3.3}$$

$(k=0, 1, \cdots, n;\ n\geqq 1)$.

Proof. We begin by noting that

$$\Delta_k(p^{(k)}) = [p_k^{(k)}(x)]^2 = \left\{k!\prod_{r=0}^{k-1}A_r\right\}^2 > 0 \qquad (k\geqq 1),$$

$$= 1 \qquad (k=1).$$

It follows from (3.2) that if $G_{n,k}=$g.l.b. $\Delta_n(p^{(k)})$ then

$$G_{n,k} \geqq \frac{A_nC_{n-1}}{A_{n-1}}G_{n-1,k}.$$

Iteration of this shows that

$$G_{n,k} \geqq \{C_{n-1}C_{n-2}\cdots C_k\}\cdot\{A_n/A_k\}\cdot G_{k,k}.$$

Since $G_{k,k}=\{k!\prod_{r=0}^{k-1}A_r\}^2$ we obtain our desired result.

One interesting special case of this theorem is when $A_n=1$. This case includes:

(i) the Hermite polynomials: $C_n = n$,

(ii) a class of generalized Hermite polynomials studied by Toscano [6]: $C_n = n + \beta$ where $\beta > -1$,

(iii) a class of polynomials studied by Carlitz [1] in connection with the special functional equation: $C_n = n^2$,

(iv) the Tchebycheff polynomials: $C_n = 1/2$.

The inequality (3) becomes for Case (i),

$$\Delta_n(H^{(k)}) \geqq (n-1)!(k!)k.$$

The corresponding inequality for other special cases can be easily computed.

We also remark here that although the derivatives of the Legendre polynomials satisfy the Turan inequality, the theorem we established above does not include this case.

4. The Laguerre polynomials are defined by the recurrence relation

$$(n+1)L_{n+1}^{(\alpha)}(x) = (-x + 2n + \alpha + 1)L_n^{(\alpha)}(x) - (n + \alpha + 1)L_{n-1}^{(\alpha)}(x)$$

$$L_0^{(\alpha)}(x) = 1, \; L_1^{(\alpha)}(x) = (1 + \alpha) - x.$$

They also satisfy

$$\frac{d}{dx} L_n^{(\alpha)}(x) = - L_{n-1}^{(\alpha+1)}(x)$$

from which we see that

$$\Delta_n\left(\frac{d^k}{dx^k} L^{(\alpha)}\right) = \Delta_{n-k}(L^{(\alpha+k)}).$$

By [5],

$$\Delta_n(L^{(\alpha)}) \geqq 0 \qquad (\alpha > -1; \; -\infty < x < \infty).$$

Hence we finally get

$$\Delta_n\left(\frac{d^k}{dx^k} L^{(\alpha)}\right) \geqq 0 \quad (\alpha > -k-1; \; -\infty < x < \infty).$$

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EXPECTATIONS FOR SUMS OF POWERS

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A sequence of independent random variables with a uniform distribution is chosen from the interval (0, 1). The process is continued until the sum of the nth powers of the chosen numbers exceeds 1. One problem that arises is to determine the expected number of such choices.

This problen has been solved by W. Weissblum [1] for $n=1$ in which case the expected number is e. In this note, we solve the general problem in terms of a definite integral and find, thereby, the asymptotic behavior as $n\to\infty$.

Heuristically, it might be reasoned, the "average" number is $1/2$ and so the expectation should be around 2^n. This is not the case and in fact it turns out that the solution which we designate by E_n satisfies $E_n\sim cn$ where c is given below. This is not really surprising since the "average" nth power number is $\int_0^1 x^n dx = 1/(n+1)$.

E_n will be given by [2]

$$E_n = 1 + f_1 + f_2 + \cdots,$$

where f_i is the probability of failure up to and including the ith trial. Geometrically, f_i will be given by the volume enclosed by

$$x_1^n + x_2^n + \cdots + x_i^n \leqq 1, \qquad x_r \geqq 0 \ (r = 1, 2, \cdots i).$$

This volume is determined immediately by an application of Dirichlet's integral ([3], p. 258) and is given by $f_i = [\Gamma(1+1/n)^i]/[\Gamma(1+i/n)]$. It follows that

$$E_n = \sum_{i=0}^{\infty} \frac{\Gamma(1+1/n)^i}{\Gamma(1+i/n)}.$$

For $n=1$, we obtain Weissblum's result $E_1 = e$.

Now using the standard contour integral ([3], p. 245)

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i}\int_c e^s s^{-z} ds$$

for $\Gamma(1+i/n)^{-1}$, shifting contours, extracting a residue, and rewriting as a real integral, we obtain for $x>0$ that

$$\sum_{i=0}^{\infty} \frac{x^i}{\Gamma(1+i/n)} = ne^{x^n} - \frac{x\sin(\pi/n)}{\pi/n}\int_0^{\infty} \frac{e^{-t^n}dt}{t^2 + x^2 - 2tx\cos(\pi/n)}.$$

E_n is equal to the above expression for $x = \Gamma(1+1/n)$.

We now find E_n, asymptotically.

Using the relation $x = \Gamma(1+1/n) = e^{-\gamma/n + O(1/n^2)}$ in the above integral, we find that

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$$\int_0^\infty \frac{e^{-t^n}dt}{t^2+x^2-2tx\cos(\pi/n)} \sim \int_{-\infty}^\infty \frac{e^{-e^u}du}{(u+\gamma)^2+\pi^2},$$

while $x^n \to e^{-\gamma}$. Consequently, $E_n \sim cn$, where

$$c = e^{e^{-\gamma}} - \int_{-\infty}^\infty \frac{e^{-e^u}du}{(u+\gamma)^2+\pi^2}.$$

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CLASSROOM NOTES

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TAYLOR'S THEOREM AND NEWTON'S METHOD

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Most texts in calculus study Newton's method long before the study of Taylor's series and Taylor's theorem. While studying the latter it may be worth while to stop to refine Newton's method and to show the relation between the two.

If we use only the first two terms of a Taylor series of a function $f(x)$ expanded about a point x_1 near a zero of $f(x)$, then $f(x)=f(x_1)+f'(x_1)(x-x_1)$. Setting $f(x)$ equal to zero we get an approximation to the root $x=x_1-f(x_1)/f'(x_1)$, which is familiar to the student as Newton's method. If now we take three terms of the Taylor series, we get an approximation

$$x = x_1 - \frac{f'(x_1) \pm \{[f'(x_1)]^2 - 2f(x_1)f''(x_1)\}^{1/2}}{f''(x_1)}$$

provided, of course, $f''(x_1) \neq 0$. While Newton's method fits a line to the graph of the function, this refinement fits a parabola with the same slope and curvature to the graph of the function. The choice of the sign is dictated by the problem. Further refinements are possible of course, but not practical.

In solving $x^4+2x^3+x^2-6x-12=0$ with a first approximation $x=2$, Newton's method yields 1.7778, the suggested refinement yields 1.7265, whereas the correct root is $\sqrt{3}=1.7321$.

EQUATIONS WITH TRIGONOMETRIC VALUES AS ROOTS

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The purpose of this note is to make readily available for classroom use the 64 irreducible polynomial equations with integral coefficients and of degree two through seven whose roots are of the form $\pm\sin y$, $\pm\cos y$, $\pm\tan y$, $\pm\cot y$, $\pm\sec y$, or $\pm\csc y$, where y is a rational number of degrees.

Of these 64 equations, 14 are quadratic, 8 are cubic, 18 are quartic, 4 are of degree five, and 20 are of degree six. There is none of degree seven. The 6 quadratics involving 30°, 45° and 60° will not be listed. The remaining 58 equations are the following 22 along with others obtained from these 22 by changing the roots to their negatives, to their reciprocals, and to their negative reciprocals:

Equation	Roots
(1) $4x^2+2x-1=0$	$\sin 18°$, $-\sin 54°$
(2) $x^2-4x+1=0$	$\tan 15°$, $\tan 75°$
(3) $x^2+2x-1=0$	$\tan 45°/2$, $-\tan 135°/2$
(4) $8x^3-6x+1=0$.	$\sin 10°$, $\sin 50°$, $-\sin 70°$
(5) $8x^3+4x^2-4x-1=0$	$\sin 90°/7$, $-\sin 270°/7$, $\sin 450°/7$
(6) $5x^4-10x^2+1=0$	$\pm\tan 18°$, $\pm\tan 54°$
(7) $16x^4-16x^2+1=0$	$\pm\sin 15°$, $\pm\sin 75°$
(8) $16x^4+8x^3-16x^2-8x+1=0$	$\sin 6°$, $-\sin 42°$, $-\sin 66°$, $\sin 78°$
(9) $x^4-4x^3-14x^2-4x+1=0$	$\tan 9°$, $-\tan 27°$, $-\tan 63°$, $\tan 81°$
(10) $16x^4-20x^2+5=0$	$\pm\sin 36°$, $\pm\sin 72°$
(11) $8x^4-8x^2+1=0$	$\pm\sin 45°/2$, $\pm\sin 135°/2$
(12) $x^4+8x^3+2x^2-8x+1=0$	$\tan 15°/2$, $\tan 75°/2$, $-\tan 105°/2$, $-\tan 165°/2$
(13) $x^4+4x^3-6x^2-4x+1=0$	$\tan 45°/4$, $-\tan 135°/4$, $\tan 225°/4$, $-\tan 315°/4$
(14) $32x^5-16x^4-32x^3+12x^2+6x-1=0$	$\sin 90°/11$, $-\sin 270°/11$, $\sin 450°/11$, $-\sin 630°/11$, $\sin 810°/11$
(15) $3x^6-27x^4+33x^2-1=0$	$\pm\tan 10°$, $\pm\tan 50°$, $\pm\tan 70°$
(16) $64x^6-96x^4+36x^2-3=0$	$\pm\sin 20°$, $\pm\sin 40°$, $\pm\sin 80°$
(17) $64x^6-112x^4+56x^2-7=0$	$\pm\sin 180°/7$, $\pm\sin 360°/7$, $\pm\sin 540°/7$
(18) $7x^6-35x^4+21x^2-1=0$	$\pm\tan 90°/7$, $\pm\tan 270°/7$, $\pm\tan 450°/7$
(19) $x^6-12x^5+3x^4+40x^3+3x^2-12x+1=0$	$\tan 5°$, $\tan 25°$, $-\tan 35°$, $-\tan 55°$, $\tan 65°$, $\tan 85°$
(20) $x^6-8x^5-13x^4+48x^3-13x^2-8x+1=0$	$\tan 45°/7$, $-\tan 135°/7$, $\tan 225°/7$, $\tan 405°/7$, $-\tan 495°/7$, $\tan 585°/7$
(21) $64x^6-32x^5-96x^4+48x^3+32x^2-16x+1=0$	$\sin 30°/7$, $\sin 150°/7$, $-\sin 330°/7$, $\sin 390°/7$, $\sin 510°/7$, $-\sin 570°/7$
(22) $64x^6+32x^5-80x^4-32x^3+24x^2+6x-1=0$	$\sin 90°/13$, $-\sin 270°/13$, $\sin 450°/13$, $-\sin 630°/13$, $\sin 810°/13$, $-\sin 990°/13$

These equations complete the list started in "Trigonometric Values that are Algebraic Numbers," Kenneth W. Wegner, *The Mathematics Teacher*, December, 1957. In that article derivations and suggested uses of equations with sines and cosines as roots were given. Here derivations of the tangent equations will be illustrated with (19).

In the identity $(3\tan^2 y-1)\tan 3y=\tan^3 y-3\tan y$, substitute $y=5°$, $65°$, and $125°$, obtaining $(3x^2-1)(2-\sqrt{3})=x^3-3x$ with roots $\tan 5°$, $\tan 65°$, and

$-\tan 55°$. Similarly, $(3x^2-1)(2+\sqrt{3})=x^3-3x$ has roots $\tan 25°$, $\tan 85°$, and $-\tan 35°$. Equation (19) is obtained by combining these two equations.

That the 64 equations referred to above are irreducible can be verified by theorems on degrees of algebraic numbers as given on page 37 of *Irrational Numbers* by Ivan Niven (No. 11 of The Carus Mathematical Monographs). That the 64 equations are the only ones of the type described can be established by a method suggested by D. H. Lehmer (this MONTHLY, vol. 40, 1933, p. 165).

A NOTE ON AN ADDITIVE PROPERTY OF NATURAL NUMBERS

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Some years ago A. Moessner drew attention to the following property of natural numbers.

Consider a set of number sequences beginning with the sequence of natural numbers. Skipping every nth number we form the sequence of partial sums. In this sequence skip every $(n-1)$th number and again form the sequence of partial sums. Again every $(n-2)$th number is skipped and so on. Then the nth sequence will be $1, 2^n, 3^n, 4^n, \cdots$.

This property is trivial for $n=2$. If $n>2$ it may be noticed that, whereas the first and the nth sequences are arithmetical progressions (of the first and nth order), the intermediate sequences are not.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$n=3$	1	3		7	12		19	27		37	48		61	75	
	1			8			27			64			125		

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
	1	3	6		11	17	24		33	43	54		67	81	96
$n=4$	1	4			15	32			65	108			175	256	
	1				16				81				256		

Several rather intricate proofs and generalizations of Moessner's statement have been given (see References). However it may be useful to show that his result can be looked upon as a straightforward consequence of Horner's algorithm applied to the polynomial x^n.

According to Horner's algorithm the coefficients of, *e.g.*, $a_0x^3+a_1x^2+a_2x+a_3$, arranged as a polynomial in $x-1$, are found in the following way.

a_0	a_1	a_2	a_3
	a_0	a_0+a_1	$a_0+a_1+a_2$
a_0	a_0+a_1	$a_0+a_1+a_2$	$a_0+a_1+a_2+a_3$
	a_0	$2a_0+a_1$	
a_0	$2a_0+a_1$	$3a_0+2a_1+a_2$	
	a_0		
a_0	$3a_0+a_1$		

From the special polynomial $x^3=x^3+0x^2+0x+0$ we arrive at $(x-1)^3+3(x-1)^2+3(x-1)+1$, from this at $(x-2)^3+6(x-2)^2+12(x-2)+8$, proceeding stepwise by means of the Horner blocks

1	0	0	0
	1	1	1
1	1	1	1
	1	2	
1	2	3	
	1		
1	3		

1	3	3	1
	1	4	7
1	4	7	8
	1	5	
1	5	12	
	1		
1	6		

1	6	12	8
	1	7	19
1	7	19	27
	1	8	
1	8	27	
	1		
1	9		

Clearly, this sequence can be prolonged ad infinitum, as the bottom figures of each block appear in the first row of the next one.

From this block sequence Moessner's sequences ($n=3$) can be derived in the following way. First suppress all *copied* figures (appearing in the copied first rows, but also in all even rows). Next write the blocks columnwise under each other. Then the left of the two arrays underneath is found; the right one is the analogue derived from the polynomial x^4,

x^3+0x^3+0x+0			
1	0	0	0
1	1	1	1
1	2	3	
1	3		
1	4	7	8
1	5	12	
1	6		
1	7	19	27
1	8	27	
1	9		
1	10	37	64

$x^4+0x^3+0x^2+0x+0$				
1	0	0	0	0
1	1	1	1	1
1	2	3	4	
1	3	6		
1	4			
1	5	11	15	16
1	6	17	32	
1	7	24		
1	8			
1	9	33	65	81

Apparently, these arrays contain Moessner's sequences ($n=3$ and $n=4$) written down vertically. According to Horner's algorithm the numbers of the last column are the constant terms of the polynomial x^n when expressed in powers of $x-i$. Hence they are the numbers i^n. This is the statement of Moessner.

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4. Ivan Paasche, Ein neuer Beweis des Moessnerschen Satzes, Sitz. Bayer. Akad. Wiss., 1953, Nr. 1, pp. 1–11.
5. ———, Z. math. u. naturw. Unterricht, 1953, VI, p. 26–28.
6. ———, Eine Verallgemeinerung des Moessnerschen Satzes, Compositio Mathematica, 1954, vol. 12, pp. 263–270.

MATHEMATICAL EDUCATION NOTES

Edited by JOHN R. MAYOR, American Association for the Advancement of Science and the University of Maryland, and JOHN A. BROWN, University of Delaware

Contributions for this department should be sent to John R. Mayor, 1515 Massachusetts Avenue, N.W., Washington 5, D. C.

CURRICULUM STUDIES IN MATHEMATICS

Because of the great effort now being given to improvement of instruction in mathematics, even mathematicians sometimes have difficulty in keeping up with what is going on. In response to numerous requests, there are listed twelve curriculum studies in mathematics, with addresses from which further information can be obtained. In the cases in which only names and addresses are given, those directing the studies have been invited to contribute longer statements for publication in this Department. Two of these appear in this issue. The others have already appeared or will be included in later issues.

The editors apologize for any omissions in this list and will be pleased to have their attention called to studies which should be included in a supplementary list.

Advanced Placement Program. The Advanced Placement Program is a continuation of the School and College Study of Admission with Advanced Standing. The Program provides descriptions of college-level courses to be given in schools and prepares examinations based on these courses. Colleges, in turn, consider for credit and advanced placement students who have taken these courses and examinations. The program is thus an instrument of cooperation which extends the educational opportunities available to able and ambitious students by coordinating effectively their work in school and college. Teachers who are setting up college-level courses should read the course descriptions in the booklet, *Advanced Placement Program Syllabus*, which may be obtained by writing to the Advanced Placement Program, College Entrance Examination Board, 425 West 117th Street, New York 27, New York.

American Society for Engineering Education. A report on the teaching of mathematics for engineers of the joint committee of the American Society for Engineering Education and the Mathematical Association of America. *Journal of Engineering Education*, vol. 45, 1955, and this MONTHLY, vol. 62, 1955, pp. 385–392.

Ball State Teachers College. For the past three years, an experimental tenth grade geometry course has been offered at the laboratory high school of Ball State Teachers College. The postulate set that is used is a modified version of the Hilbert postulates. Plans for this year call for continuation of this course in the laboratory school and several high schools near the College. Work has also been started on a ninth grade algebra course. For information, write to Professor Charles F. Brumfiel, Ball State Teachers College, Muncie, Indiana.

University of Chicago. Since the founding of the College at the University of Chicago, the mathematics department of the College has been concerned with the development of a modern and basic course in mathematics for general education. The course now in use, which has gone through frequent revision, involves many of the ideas with which other of the curriculum groups are now working. The experience at the University of Chicago has been of great value and will continue to be of value to all efforts to modernize the mathematics curriculum. For information write to Professor A. L. Putnam, Box 23, Eckhart Hall, University of Chicago, Chicago 37, Illinois.

Commission on Mathematics. Dr. Robert E. K. Rourke, Executive Director, Commission on Mathematics, 425 West 117th Street, New York 27, New York.

Committee on the Undergraduate Program. For about five years the Mathematical Association of America has sponsored a Committee on the Undergraduate Program. The Committee has considered ways (1) to bring to freshmen the calculus, (2) to introduce to them set notions and probability theory, (3) and to do this without imposing unrealistic demands on students with two and one-half years of mathematics in high school. The Committee has designed two types of sophomore courses. One of these is for physical science and engineering majors, and the other is intended to answer increasing demands from the biological and social sciences. Copies of Collected Reports of the C.U.P. may be obtained from the Mathematical Association of America, University of Buffalo, Buffalo, New York. The Committee was discharged on September 1, 1958, but the Association is making plans to continue this work.*

University of Illinois Committee on School Mathematics. Dr. Max Beberman, UICSM, University High School, Urbana, Illinois.

University of Maryland Mathematics Project (Junior High School). Dr. John R. Mayor, Director of Education, American Association for the Advancement of Science, 1515 Massachusetts Avenue, N.W. Washington 5, D.C.

National Council of Teachers of Mathematics Secondary School Curriculum Committee. The Secondary School Curriculum Committee of the National Council of Teachers of Mathematics has been organized with ten subcommittees, each of which is expected to issue a preliminary report in the spring of 1959. The Committee in giving its attention to the mathematics program for *all* students, grades 7 through 12, is in a strategic position to assist all of the current curriculum investigations, to interpret for teachers and schools recommendations of other groups, and to bring about acceptance by schools and teachers of sound proposals of all other groups. For information, write to Mr. Frank B. Allen, Lyons Township High School, La Grange, Illinois.

Oklahoma State Committee for the Improvement of Mathematics Instruction. Dr. James H. Zant, Oklahoma State University, Stillwater, Oklahoma.

School Mathematics Study Group. Dr. E. G. Begle, School Mathematics Study Group, Drawer 2502A, Yale Station, Yale University, New Haven, Connecticut.

Social Science Research Council Committee on Mathematics in Social Science Research. This Committee was appointed by the Council in 1958 as the successor to its former Committee on Mathematical Training of Social Scientists. The former Committee planned and conducted five summer institutes; it prepared a statement recommending policies for the mathematical training of social scientists (ITEMS, June 1955, p. 13), and aided in the preparation of other materials. For information and reprints write to Social Science Research Council, 230 Park Avenue, New York 17, New York.

MATHEMATICS CURRICULUM STUDIES IN OKLAHOMA

JAMES H. ZANT, Oklahoma State University

Curriculum revision in mathematics has been approached in Oklahoma with the belief that the development of an overall mathematics program from the kindergarten through grade 12 is desirable. The program should be designed to insure for all students mathematical competency and an appreciation of the role of mathematics from a modern aspect, to enable them to meet the challenge

* A conference on the work of the CUP was held in Washington, D. C. November 15–16, 1958. A report of the conference will appear in an early issue of the MONTHLY.

of living adequately in a highly-complex and technological age. The stimulus for the program as it has developed came from the Oklahoma Curriculum Improvement Commission of the State Department of Education with the active interest and financial support of the Frontiers of Science Foundation of Oklahoma, Inc. and from public-minded school administrators from over the state.

In the summer of 1957 a one-month Mathematics Workshop was held at the Oklahoma State University. Participants consisted of a group of able teachers of mathematics in the Oklahoma schools, who were assisted by a staff of competent mathematicians and mathematics educators from within the state and from the outside. This group developed a 94 page bulletin* for the purpose of helping all teachers in their efforts to improve instruction in mathematics from grades one through twelve.

It was recognized that, though the general education of young men and women will be neglected unless they have a thorough understanding of the principles of mathematics, perhaps no area in the curriculum has been less subject to modification and change. Curriculum reorganization, which involves the introduction of new and unfamiliar concepts, must be done on a very broad base if results are to be achieved with any degree of rapidity. This group of teachers and mathematicians used this broad approach in a first attempt to open the way to bring the mathematics curriculum up to date and to provide teachers at all levels with material that they can use to improve their own competence. It was hoped that through their efforts there would be introduced into programs of study new approaches to old subject matter and, perhaps more important, this would encourage the introduction of new subject matter of great importance in the modern world but relatively lacking in the traditional approach to mathematics.

A total of 15,000 copies of the above-mentioned bulletin were printed and distributed widely over the state by the State Department of Education. It was soon apparent, however, that without adequate leadership the efforts of the Mathematics Workshop would not be truly effective over the state as a whole. Hence a committee appointed by the Oklahoma Curriculum Improvement Commission, called the State Steering Committee for Mathematics, began discussing future plans. In order to consolidate state leadership and to initiate a permanent program for improving the state mathematics curriculum, the Commission, following the suggestion of the Steering Committee and again with the active financial support of the Frontiers of Science Foundation of Oklahoma, Inc. and the State Department of Education, sponsored a Leadership Conference on the Improvement of Mathematics Teaching in the Schools of Oklahoma, March 19–22, 1958, at Oklahoma City.

The participants in the Leadership Conference were from the colleges and universities, public and private, engaged in teacher training in the state and included a small number, probably too few, of teachers and administrators

* The Improvement of the Teaching of Mathematics. The Oklahoma State Department of Education, Oliver Hodge, Superintendent. $1.00.

from the public schools. Fourteen of the seventeen colleges and universities were represented, in most cases by two staff members, one from mathematics and one from education. There were 35 participants. The activities of the conference consisted of lectures on various phases of mathematics and mathematics education by the consultants and leaders in the state. Discussions sharpened and clarified points made by the various speakers and at the end the group arrived at definite conclusions and a plan for action in the state.* The "plan for action" included a recommendation to the Oklahoma Curriculum Improvement Commission that the Commission appoint a *State Committee for the Improvement of Mathematics Instruction* including a State Supervisor for Mathematics Education and approximately 20 members consisting of college or university mathematicians, professional education staff members, and elementary and secondary school teachers and supervisors. Further developments of a program in improving mathematics teaching in Oklahoma will be the responsibility of the State Committee working through the Oklahoma Curriculum Improvement Commission.

The State Committee has been appointed with James H. Zant as the Chairman and the program for the future activities in this area is now being completed. It is expected that this will include plans for organizing study groups and subcommittees in various areas; definite plans for writing teaching materials, first in the form of teaching units and eventually in the form of courses for grades and areas; making a definite effort to get information about the need and methods of a new program in mathematics before the teachers and administrators of the state; plans for retraining teachers now in service; and for the reorganization of pre-service training of our elementary and secondary mathematics teachers.

THE UNIVERSITY OF MARYLAND MATHEMATICS PROJECT

M. L. KEEDY, University of Maryland

The University of Maryland Mathematics Project (Junior High School), made possible by a grant from the Carnegie Corporation of New York, is now beginning the second of its three years. The study is being directed by John R. Mayor, with the author as his associate. An advisory committee, which assists in the formulation of policy, represents the areas of mathematics, science, engineering, psychology and education within the university, the U. S. Office of Education, the Maryland State Department of Education, and the public school systems near the University. The four major school systems in the Washington, D. C., area—Prince Georges and Montgomery Counties, Maryland; Arlington County, Virginia; and the District of Columbia—are cooperating in the study, with some twenty-five junior high school mathematics teachers participating. These teachers meet weekly at the University of Maryland, to in-

* A 126-page report consisting of the papers read at the Conference and the final recommendations to the Commission was multilithed for distribution within the state.

crease their knowledge of modern mathematical concepts and of recent advances in the theories of learning. They also assist with preparation and revision of teaching materials, conduct interviews of children, and teach newly-prepared materials in their own classes.

The primary goal of the project is to determine maturity levels at which certain mathematical concepts can be appropriately taught, and to prepare materials for a teaching sequence in grades seven and eight which is mathematically and psychologically sound, and appropriate to modern-day needs. Results of the study are being made available to curriculum planning groups across the country, and increasing cooperation with such groups and with other persons engaged in experimental work in mathematics curriculum is being developed.

During the summer of 1958 the staff prepared the first portion of an experimental seventh grade course, on the basis of experience gained in the first year of the study, in addition to conducting a four-week National Science Foundation Institute in conjunction. The 44 Institute participants, representing a wide geographic area, studied mathematics and studied and assisted with preparation of the experimental course materials, in addition to observing an experimental class of seventh grade children.

In the academic year 1958–59 the experimental seventh grade course is being taught in about 25 schools in the greater Washington area and in some others as well. A psychologist has joined the staff for the purpose of directing an evaluation of the experimental course. The teachers involved will continue in a weekly seminar at the University of Maryland, assisting with revision of the experimental course and beginning to develop materials for an eighth grade course, to be taught the following year.

In the experimental seventh grade course, arithmetic and algebra are not distinguished, but are merged by considering *properties of numbers*. The concept of a *mathematical system* is developed and used thereafter as applicable. Unifying concepts are given stress, vocabulary is simplified where possible, and linguistic precision is emphasized where mathematical content is involved. Applied problems of types ordinarily taught in grade seven are included in the exercises, but the first emphasis is on mathematical understanding. The course is designed to include procedures in which students reason both inductively and deductively, although the deduction is largely of an informal nature.

REACTIONS TO THE BOWLING GREEN CONFERENCE*

W. Norman Smith, University of Wyoming

To report on a meeting of the size and complexity of the Bowling Green Conference accurately, briefly and objectively is impossible. The best I can do

* This is the third statement by a mathematician who participated in the 1958 Annual Conference of the National Commission for Teacher Education and Professional Standards. Other statements by F. A. Ficken and Paul C. Rosenbloom appeared in this Department in November, 1958.

is to list a few random impressions, colored by the discussion group of which I was a member, as well as by my personal views.

The most striking, as well as the most encouraging, impression was that of the apparent recognition by delegates from the entire educational spectrum of the importance of subject-matter preparation in the training of teachers. I was assured by one delegate that this was a radical shift—that five years ago any mention of subject matter would have been squelched by the shibboleth, "You don't teach mathematics, you teach boys and girls."

Just how the prospective teacher was to find the time for more intensive work was a moot point. There was a general feeling that five years of college work should be a minimum, but many of the delegates felt that this was impractical as an immediate goal. There seemed to be no willingness on the part of the professional educators to reduce the number of required hours in education, nor did the teachers themselves suggest that this should be done, although all agreed that a certain amount of pruning in the educational vineyard was desirable.

Another encouraging aspect was the realization that the education of teachers was a joint project requiring close cooperation between the liberal arts departments and the professional educators. It was, however, obvious that not all of the heat of conflict between these two groups had been dissipated.

I am confident that the Bowling Green Conference will lend impetus to a movement already underway in Wyoming for closer cooperation between the teachers in the state, the College of Education, and the College of Arts and Sciences.

NATIONAL DEFENSE EDUCATION ACT OF 1958

This Act, passed in the closing days of the 85th Congress, authorizes the expenditure of funds through the Office of Education of the Department of Health, Education, and Welfare under the headings: Loans to Students; Financial Assistance for Strengthening Science, Mathematics, and Modern Foreign Language Instruction; Fellowships; Guidance, Counseling, and Testing; Language Development; Research in Television and Motion Pictures; Area Vocational Education Programs.

Mathematicians should become familiar with the plans for implementation of this Act in their states, especially in relation to the title on Financial Assistance for Strengthening Science, Mathematics, and Foreign Language Instruction.

Inquiries might be addressed to state departments of education or to Dr. Kenneth E. Brown, U. S. Office of Education.

CONTINENTAL CLASSROOM IN PHYSICS*

CONTINENTAL CLASSROOM—the TV course for college credit in Atomic Age Physics launched October 6 over the National Broadcasting Company nationwide network—is proving to be a "hit." Over 250 colleges and universities across the country are offering the course for credit. In addition, at least 20,000 engineers, technicians, homemakers,

* Submitted upon request by The American Association of Colleges for Teacher Education, Office of the National Coordinator of Atomic Age Physics. For further information: Edwin P. Adkins or Hope C. Corso, In Care of National Broadcasting Company, 30 Rockefeller Plaza, New York 20, New York.

gifted high school students, career-Army personnel and others are viewing the program to update themselves in modern-day science. Teaching the course is Dr. Harvey E. White, professor of physics at the University of California and consultant to the Atomic Energy Commission. Guest lecturers include noted physicists and scientists from leading U. S. institutions of higher education.

Supervising the over-all effort—for the American Association of Colleges for Teacher Education—is Dr. Edwin P. Adkins, on leave from New York State University College for Teachers in Albany where he is director of education. Financial support for the program has been contributed by the Ford Foundation, the Fund for the Advancement of Education, and by industry: Bell Telephone System, the California Oil Company, General Foods Fund, International Business Machines Corporation, Pittsburgh Plate Glass Foundation and United States Steel.

The second semester on Nuclear Physics, commencing February 11, promises to attract an even larger number of viewers than this semester's offering on Basic Physics. The program—telecast at 6:30 a.m., local time—will end June 5.

CURRENT ITEMS

The complete report on *Science and Math* in the *NASSP Bulletin*, Sept. 1958, pp. 5–12, is available in reprint form on request as long as the supply lasts. For a free copy of *The Place of Science and Mathematics in the Comprehensive Secondary-School Program*, write to *The Spotlight*, 1201 Sixteenth Street, N.W., Washington 6, D.C.

Science and mathematics receive special emphasis in a plan to give four different diplomas which has recently been approved by the Indianapolis Board of Education. The four diplomas do not simply reflect the kind of study undertaken, for they also take into account how well the students do.

A National survey of science and mathematics teachers in American public high schools has been started by the U. S. Office of Education and will be ready for distribution around the first of the year. This survey is the most recent of similar surveys published in 1955 and 1957.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1346. *Proposed by J. M. Gandhi, Belgaum, India*

Prove that if p is a prime then $\binom{2p}{p} \equiv 2, \bmod p$.

E 1347. *Proposed by V. F. Ivanoff, San Carlos, California*

Prove that $\sum_{i=0}^{n} \binom{n}{i} F_{n-i} = F_{2n}$, where $F_0, F_1, \cdots, F_n$ is any set of $n+1$ consecutive Fibonacci numbers.

E. 1348. *Proposed by M. S. Klamkin and Raphael Miller, AVCO Research and Development, Lawrence, Mass.*

Find the locus of the centroids of all equilateral triangles inscribed in an ellipse.

E 1349. *Proposed by P. L. Chessin, University of Maryland*

Consider the $n \times n$ matrix $[a_{ij}]$ where $a_{11} = \cos\theta$, $a_{ii} = 2\cos\theta$ $(i = 2, \cdots, n)$, $a_{i,i+1} = a_{i+1,i} = 1$ $(i = 1, \cdots, n-1)$, and all other elements are zero. Show that $|a_{ij}| = \cos n\theta$.

E 1350. *Proposed by N. A. Court, University of Oklahoma*

(a) The tangents to the ninepoint circle of a triangle T at the midpoints of the sides of T form a triangle homothetic to the orthic triangle of T. (b) The homothetic center of the two triangles is a point on the Euler line of T.

SOLUTIONS

Existence of a Unit Element

E 1316 [1958, 365]. *Proposed by R. C. Buck, University of Wisconsin*

Let S be a set of elements with an associative multiplication. Suppose that S has a special element u with the property that u is a left and a right divisor of every element in S. Does S have to possess a unit?

Solution by J. V. Whittaker, University of British Columbia. For any $x \in S$, let $yu = x = uz$ and $su = u = ut$. Then $sx = s(uz) = (su)z = uz = x$ and, similarly, $xt = x$. Thus $st = s = t$ is a unit of S.

Also solved by R. G. Albert, M. D. Anderson and Jerry Bebernes (jointly), Lawrence Arnold, A. Bager, Peter Beisswanger, L. P. Belluce, M. P. Berenson, J. L. Brenner, J. L. Brown, Jr., P. L. Chessin, R. M. Conkling, H. M. Farkas, N. J. Fine, Fred Galvin, W. V. Gamzon, Virginia S. Hanly, J. Hooley, A. F. Kaupe, Jr., Irving Kay, J. M. Kingston, Joe Lipman, R. T. J. Mahoney, D. C. B. Marsh, J. S. Moore, Jr., F. D. Parker, D. S. Passman, W. J. Pervin, John Rainwater, Theodore Reiss, Azriel Rosenfeld, Jack Silver, Paul Slepian, J. W. Smith, Anthony Trampus, and the proposer.

A Tetrahedron and a Sphere

E 1317 [1958, 365]. *Proposed by N. A. Court, University of Oklahoma*

Let $(T) = DABC$ be a tetrahedron and let (M) be a sphere having its center M on the axis of the circle ABC. If one and only one point is taken in each pair of points determined by (M) on the edges of (T) issued from the vertex D, show that the $2^3 = 8$ planes thus determined are cut by the plane ABC along four pairs of isotomic transversals of the triangle ABC.

Solution by the Proposer. 1. Let

(a) $$U, U'; V, V'; W, W$$

be the traces of (M) on the edges DA, DB, DC, respectively. The points V, V',

W, W' determine a complete quadrangle (q) inscribed in the circle (M_a) in which plane DBC cuts (M). The four pairs of points

$$(I) \qquad P, P'; X, X'; B, C; S, S'$$

in which line BC cuts the three pairs of opposite sides VW, $V'W'$; VW', $V'W$; VV', WW' of (q) and the circle (M_a), belong to the same involution, by Desargues' theorem [see, for inst., L. Cremona, *Projective Geometry*, p. 143, art. 183].

2. The points of intersection S, S' of (M_a) with line BC are also the traces on BC of the sphere (M). These points therefore lie also in the circle (M_d) in which (M) is cut by plane ABC. Now (M_d) is concentric with circle ABC, since by assumption M lies on the axis of circle ABC. Hence the two segments SS' and BC have the same midpoint, say A'.

The two pairs of points B, C; S, S' determine the symmetrical involution having A' and the point at infinity of BC as the double points, or, what is the same thing, the involution of isotomic points on side BC of triangle ABC. On the other hand, these two pairs of points also belong to the involution (I) considered in part 1. Hence (I) is identical with the isotomic involution. Consequently, the two pairs of lines VW, $V'W'$; VW', $V'W$ determine on BC two pairs of isotmic points (*cf.* Nathan Altshiller-Court, *College Geometry*, 2nd ed., p. 158, ex. 7). We similarly treat the faces DCA, DAB of (T).

3. Consider any two of the eight planes determined by the points (a) and involving all six points, say UVW and $U'V'W'$. The three pairs of lines VW, $V'W'$; WU, $W'U'$; UV, $U'V'$ determine on the three transversals BC, CA, AB, respectively, three pairs of isotomic points P, P'; Q, Q'; R, R' (part 2). The triads of points PQR, $P'Q'R'$ lie in the planes UVW, $U'V'W'$, respectively, and all six lie in the plane ABC. Hence the points of each triad are collinear, and the two lines PQR, $P'Q'R'$ are two isotomic transversals of triangle ABC. We may similarly treat the other three pairs of analogous planes. Hence the proposition.

The reader may consider the cases where the sphere (M) passes through the vertex D or through the circle ABC and formulate the corresponding results.

A Theorem on Permutations

E 1318 [1958, 366]. *Proposed by Gordon Raisbeck, Bell Telephone Laboratories, Inc.*

A. A. Mullin has proved (this MONTHLY [1957, p. 669]) that

$$\Phi(n) = \sum_{r=0}^{n} {}_nP_r = \sum_{r=0}^{n} \frac{n!}{(n-r)!} \sim (n!)e$$

for large n, where $\sim$ denotes asymptotic equality. Prove that

$$\Phi(n) = [(n!)e], \qquad n \geq 1,$$

where $[x]$ denotes the largest integer not greater than x.

I. *Solution by W. J. Blundon, Memorial University of Newfoundland.* We have

$$(n!)e - \Phi(n) = n!\{1/(n+1)! + 1/(n+2)! + 1/(n+3)! + \cdots\}$$
$$= 1/(n+1) + 1/(n+1)(n+2) + 1/(n+1)(n+2)(n+3) + \cdots.$$

This expression is clearly positive, but is less than

$$(n+1)^{-1} + (n+1)^{-2} + (n+1)^{-3} + \cdots = 1/n \leqq 1.$$

Since $\Phi(n)$ is an integer, it follows that $\Phi(n) = [(n!)e]$.

II. *Solution by J. D. E. Konhauser, State College, Pa.* Since $\Phi(n)$ is the total number of permutations of n objects, this problem is equivalent to the solved problem E 1186 [1956, 343].

Also solved by R. G. Albert, Lawrence Arnold, Peter Beisswanger, J. L. Brenner, J. L. Brown, Jr., P. L. Chessin, E. L. Ellis and D. L. Muench (jointly), William Faris, N. J. Fine, Fred Galvin, Lawrence Glasser, L. D. Goldberg, Michael Goldberg, Emil Grosswald, J. H. Hodges, Vern Hoggatt, Norbert Kaufman, A. F. Kaupe, Jr., John Kelley and W. E. Lawrence (jointly), Morton Kupperman, Joe Lipman, R. T. J. Mahoney, D. C. B. Marsh, Clifford Marshall, L. C. Marshall, Leo Moser, T. F. Mulcrone, S.J., A. A. Mullin, J. B. Muskat, F. D. Parker, D. S. Passman, C. F. Pinzka, D. A. Robinson, Azriel Rosenfeld, R. E. Shafer, H. K. Shepard, D. L. Smith, D. P. Thompson, L. K. Williams, David Zeitlin, and the proposer.

A Quiz Contestant

E 1319 [1958, 366]. *Proposed by S. W. Golomb, California Institute of Technology*

A quiz contestant selects a category containing n questions, k of which are too difficult for him. The questions are selected from the category at random, and the contestant continues answering until he misses a question. What is the probability that he will miss on the ath question?

Solution by C. F. Pinzka, University of Cincinnati. The probability that the contestant answers the first $a-1$ questions correctly is

$$\binom{n-k}{a-1} \Big/ \binom{n}{a-1}$$

and the probability that he then misses the ath question is $k/(n-a+1)$. The probability that these events occur in succession is

$$\frac{\binom{n-k}{a-1}}{\binom{n}{a-1}} \cdot \frac{k}{n-a+1} = \frac{\binom{n-a}{k-1}}{\binom{n}{k}} = \frac{(n-k)!(n-a)!k}{n!(n-k-a+1)!}.$$

Also solved by D. S. Adorno, R. G. Albert, Peter Beisswanger, A. P. Boblétt, Julian Braun, D. A. Breault, J. L. Brown, Jr., W. J. Cahill, P. L. Chessin, A. G. Clark, R. M. Conkling, E. L. Ellis and D. L. Muench (jointly), William Faris, N. J. Fine, Fred Galvin, L. D. Goldberg, Michael Goldberg, Edwin Goldfarb and R. W. Simister (jointly), A. G. Grace, Jr., R. E. Greenwood, J. H. Hodges, A. R. Hyde, Elaine Johnson, Irwin Kabus, Norbert Kaufman, A. F. Kaupe, Jr., D. A. Kearns, J. D. E. Konhauser, Sam Kravitz, W. H. Kruskal, W. E. Lawrence, Joe Lipman, Peter McManus, D. C. B. Marsh, Helen Marston, Leo Moser, T. H. Mott, Jr., J. B. Muskat, C. S. Ogilvy, C. A. Reiher, Azriel Rosenfeld, R. E. Shafer, Jack Silver, Paul Slepian, R. H. Wilson, Jr., David Zeitlin, and the proposer.

The problem was located in W. Feller, *Theory of Probability*, vol. 1, prob. 12, p. 59 (1st ed.) or prob. 22, p. 60 (2nd ed.).

Composite Values of a Polynomial

E 1320 [1958, 366]. *Proposed by G. S. Stoller, Polytechnic Institute of Brooklyn*

(a) Let $P(x)$ be any polynomial in x with integral coefficients. Prove that there exists an infinite number of integers t such that $P(t)$, $P(t+1)$, $\cdots$, $P(t+m)$ are all composite for any given positive integer m.

(b) Let $P(x)=x^2+1$. Find a value of t satisfying part (a) for $m=5$.

Solution by Virginia S. Hanly, North American Aviation, Columbus, Ohio. (a) Let k be arbitrary such that $|P(k)|>1$ and $x \geqq k$ implies $|P(x)|$ monotone increasing. Let u be the least common multiple of the numbers $|P(k)|$, $|P(k+1)|$, $\cdots$, $|P(k+m)|$. We set $t=u+k$. Consider $P(t+i)$ for $i=0, 1, 2, \cdots, m$. We have $P(t+i) \equiv P(k+i) \bmod u$. Thus $P(k+i)$ is a proper divisor of $P(t+i)$ so that $P(t+i)$ is composite.

(b) In the case of the function $P(x)=x^2+1$ with $m=5$ we may choose $k=1$ to obtain the value l.c.m. $(2, 5, 10, 17, 26, 37)+1=81{,}771$ for t.

Also solved by R. G. Albert, W. J. Blundon, N. J. Fine, Fred Galvin, Sidney Kravitz, Joe Lipman, D. C. B. Marsh, Leo Moser, D. S. Passman, Jeff Scargle, R. E. Shafer, Jack Silver, W. A. Veech, and the proposer.

None of the methods developed for determining t in part (a) led to the least value of t, namely $t=27$, for part (b).

Moser pointed out that the result is an immediate consequence of the following theorem of H. Heilbronn (Über die Verteilung der Primzahlen in Polynomen, *Mathematische Annalen* 104 (1931), pp. 794–799). If $f(x)$ is a nonconstant integer-valued polynomial, then there exists a constant c such that for every m the number of primes in the set $f(1), f(2), \cdots, f(m)$ is less than $cm/\log m$.

AN INTERESTING PYTHAGOREAN TRIANGLE

Victor Thébault notes an interesting Pythagorean triangle in which the two perpendicular sides are integers having the same digits in reverse order, *viz.*, 88209 and 90288, the hypothenuse being 126225.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4822. *Proposed by J. deGroot, University of Amsterdam, Netherlands*

Prove that the additive group R of rational numbers is (up to isomorphisms) the only group satisfying the following conditions: (1) R is abelian, (2) R is infinite, and (3) every endomorphism (that is a homomorphic mapping of R in itself) is either an automorphism or a mapping on the null element.

4823. *Proposed by G. Matthews, St. Dunstan's College, Catford, England*

Let X be a lower-semi-matrix whose elements x_{ij} $(i, j=1, 2, 3, \cdots)$ are independent variables if $j \leqq i$ and zero if $j>i$, and let D be the differential operator matrix defined by $D_{ij}=\partial/\partial x_{ij}$ if $j<i$, $D_{ij}=0$ if $j>i$ and $D_{ii}=\sum_{k=1}^{i} \partial/\partial x_{kk}$. Prove that $D(X^2)=2X$, and hence by induction that $D(X^n)=nX^{n-1}$, where n is any positive integer greater than 2.

4824. *Proposed by D. J. Newman, AVCO Research and Development, Wilmington, Mass.*

Let $ABCD$ be a rectangle, $AB=1$, $BC=2$. Suppose that it is conformally mapped onto the upper half plane. If $A \to a$, $B \to b$, $C \to c$, $D \to d$, exhibit an elementary relation between a, b, c, d.

4825. *Proposed by J. J. Schäffer, University of Uruguay*

Prove that in any n-dimensional (real or complex) Euclidean space, $n>1$, and indeed in any Hilbert space, every point x, $\|x\| \leqq 1$ may be written $x = \sum_{i=1}^{m} f_i(x)$, where $f_1(x), \cdots, f_m(x)$ is a fixed finite set of functions which are defined and uniformly continuous in $\|x\| \leqq 1$ and satisfy $\|f_i(x)\|=1$ for $i=1, \cdots, m$ and for all $\|x\| \leqq 1$.

From this result it follows at once that if $\mathfrak{C}$ is the Banach space of all bounded continuous functions $\phi(t)$ of a real variable t with values in any Hilbert space of dimension >1 (with the norm of the supremum), the set of $\phi(t) \in \mathfrak{C}$ with $\|\phi(t)\|=1$ identically contains a linear basis of $\mathfrak{C}$.

4826. *Proposed by M. S. Klamkin and L. A. Shepp, AVCO Research and Development, Wilmington, Mass.*

If $\phi(x)=x/1^2-x^3/3^2+x^5/5^2-x^7/7^2+\cdots$, express $\phi(1)$ in terms of $\phi(2-\sqrt{3})$, thus obtaining a more rapidly converging expansion.

4827. *Proposed by J. Gallego-Diaz, Vanderbilt University*

If the development of the function $y=f(x)$ is given by

$$y = a_1x + a_3x^3 + a_5x^5 + \cdots,$$

find a function y knowing that if we invert the series we get

$$x = a_1y - a_3y^3 + a_5y^5 - a_7y^7 + \cdots.$$

SOLUTIONS

Maximal Ideals in a Ring

4761 [1957, 676]. *Proposed by Alfredo Jones, Institute of Mathematics and Statistics, Montevideo, Uruguay*

In a ring with identity and with proper ideals, there always exist maximal ideals. Is the statement true for rings with a nontrivial multiplication and with no identity?

Solution by E. A. Walker, New Mexico A & M State College. Let R be the rational numbers. Define multiplication in the additive group $R\oplus R$ by $(r, s)\cdot(r', s')=(0, rr')$. With this multiplication, $R\oplus R$ becomes a ring with proper ideals, nontrivial multiplication, and no identity. Let I be a proper ideal in this ring. The set R_1 of first coordinates of the elements of I is not R, because then I would be $R\oplus R$. Let R_2 be a proper subgroup of R properly containing R_1. Then $R_2\oplus R$ is a proper ideal properly containing I. Thus the anwer is no.

Also solved by E. R. Gentile, K. G. Wolfson, and the proposer.

Editorial Note. The solver felt that the reader will be aware that the additive group R has no maximal subgroups (whence there must exist an R_2 as stated above.) However, at the Editor's request, he supplies the following proof.

Let R_1 be a proper subgroup of R. Let $a\in R$, $a\notin R_1$. Some non-zero multiple na is in R_1. Let R_2 be the subgroup of R generated by R_1 and a. If $a/n\in R_2$ then $a/n=r_1+ka$, $r_1\in R_1$, k an integer. Therefore $a=nr_1+kna$. But $nr_1\in R_1$, $kna\in R_1$. Hence $a\in R_1$, and this contradiction proves $a/n\notin R_2$. Hence $R_1<R_2<R$.

An Algebraic Identity

4778 [1958, 211]. *Proposed by R. C. Lyness, Preston, England*

Given $f(r)=\alpha^r(\beta-\gamma)+\beta^r(\gamma-\alpha)+\gamma^r(\alpha-\beta)$ in which α, β, γ are nonzero and distinct. If n is a positive integer and $f(n+1)=0$, prove that

$$f(n+2)f(n) = \alpha^n\beta^n\gamma^n(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)f(-n).$$

Solution by D. C. B. Marsh, Colorado School of Mines. From the product rule for determinants, the following identity is evident:

$$\begin{vmatrix} f(n+2) & f(n+1) & \gamma^{n+1} \\ f(n+1) & f(n) & \gamma^n \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} \alpha^{n+1} & \beta^{n+1} & \gamma^{n+1} \\ \alpha^n & \beta^n & \gamma^n \\ 1 & 1 & 1 \end{vmatrix} \cdot \begin{vmatrix} \alpha(\beta-\gamma) & (\beta-\gamma) & 0 \\ \beta(\gamma-\alpha) & (\gamma-\alpha) & 0 \\ \gamma(\alpha-\beta) & (\alpha-\beta) & 1 \end{vmatrix}.$$

Replacing each by its expansion, one obtains

$$f(n+2)f(n) - f^2(n+1)$$
$$= \{\beta^n\gamma^n(\beta-\gamma) + \gamma^n\alpha^n(\gamma-\alpha) + \alpha^n\beta^n(\alpha-\beta)\}\{(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)\}$$
$$= \alpha^n\beta^n\gamma^n(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)\cdot f(-n).$$

Setting $f(n+1)=0$ gives the announced result. There need be no restriction on n.

Also solved by A. H. Aheart, A. P. Boblétt, L. Carlitz, A. E. Danese, E. S. Eby, E. L. Ellis and D. L. Muench, A. B. Farnell, J. W. Haake, J. H. Hodges, R. D. James, Norbert Kaufman, Irving Kay, J. D. E. Konhauser, A. E. Landry, J. A. Larrivee, Gerald Leibowitz, O. E. Lewis, Joe Lipman, Yoshio Matsuoka, M. F. Neuts, Kyu Sam Park, F. D. Parker, Paul Payette, F. W. Ponting, Siya Ram, Benjamin Sapolsky, Marlow Sholander, Arnold Singer, T. H. Slook, Robert Spira, Chih-yi Wang, Harry Weingarten, David Zeitlin, and the proposer.

Total Variation

4779 [1958, 211]. *Proposed by Solomon Leader, Rutgers University*

Let $f(x) = |x-k|$ for $k-\frac{1}{2} \leqq x \leqq k+\frac{1}{2}$, where k runs through the integers, and

$$g_n(x) = \sum_{\nu=0}^{n-1} 10^{-\nu} f(10^\nu x).$$

($\lim_{n\to\infty} g_n(x)$ is Van der Waarden's example of a continuous, nowhere differentiable function.) Let $\|g_n\|$ be the total variation of g_n on the interval (0, 1). Find $\lim_{n\to\infty} n^{-1/2}\|g_n\|$.

Solution by the proposer. From the definition of $f(x)$ we have for every integer k:

$$f'(x) = \begin{cases} +1 & \text{for } k < x < k+\frac{1}{2} \\ -1 & \text{for } k+\frac{1}{2} < x < k+1. \end{cases}$$

Thus

$$f'(10^\nu x) = \begin{cases} +1 & \text{for } k/10^\nu < x < k/10^\nu + 5/10^{\nu+1} \\ -1 & \text{for } k/10^\nu + 5/10^{\nu+1} < x < (k+1)/10^\nu. \end{cases}$$

For x in (0, 1) and not a decimal fraction of order $\nu+1$, consider the decimal expansion of x, $x = .x_1x_2\cdots x_\nu x_{\nu+1}\cdots$. Then

$$f'(10^\nu x) = \begin{cases} +1 & \text{for } x_{\nu+1} = 0, 1, 2, 3, 4 \\ -1 & \text{for } x_{\nu+1} = 5, 6, 7, 8, 9. \end{cases}$$

For x not a decimal fraction of order n we have

$$g_n'(x) = \sum_{\nu=0}^{n-1} f'(10^\nu x).$$

Let m be the number of digits less than 5 occurring in the first n places of the decimal expansion of x. Then $g_n'(x) = m-(n-m) = 2m-n$. Now there are

$\binom{n}{m}5^n$ decimal fractions of the form $.x_1x_2 \cdots x_n$ having m digits less than 5 and $n-m$ digits equal to 5 or more. For each such decimal fraction we have for $.x_1x_2 \cdots x_n < x < .x_1x_2 \cdots x_n999 \cdots$, $g_n'(x) = 2m-n$. Thus $g_n'(x) = 2m-n$ over $\binom{n}{m}5^n$ intervals of length $1/10^n$, giving a total length $\binom{n}{m}/2^n$.

Now $\|g_n\| = \int_0^1 |g_n'(x)|\,dx$. So

$$\|g_n\| = \sum_{m=0}^{n} |2m-n| \binom{n}{m} \Big/ 2^n$$

which is just twice the mean deviation of the number m of heads in n tosses of a coin. The expectation of m is $n/2$ and the standard deviation of m is $\sqrt{n}/2$. Since

$$\frac{1}{\sqrt{n}}\|g_n\| = \sum_{m=0}^{n} \left|\frac{m-n/2}{\sqrt{n}/2}\right| \binom{n}{m} \frac{1}{2^n}$$

and the binomial distribution converges to the normal distribution, we have

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}}\|g_n\| = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |x|\, e^{-x^2/2}dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^{\infty} xe^{-x^2/2}dx = \sqrt{\frac{2}{\pi}}.$$

Also solved by Robert Breusch.

Equiproduct Points

4780 [1958, 212]. *Proposed by M. S. Klamkin and D. J. Newman, AVCO Research and Development, Wilmington, Mass.*

An equiproduct point of a curve is defined to be a point such that the product of the two segments of any chord through the point is constant. (1) Show that if every point inside a curve is equiproduct, the curve must be a circle. (2) What is the maximum number of equiproduct points a noncircular oval can have?

Editorial Note. The problem is not new. For solution and discussion see problem E 705 [1946, 395] and [1947, 164]. In the latter reference will be found a discussion of a paper by K. Yanagihara in the Tohoku Mathematical Journal (1917) touching on the same problem, together with analogous theorems for three dimensional space. The principal result is the theorem: *A convex closed curve having two distinct interior equiproduct points is a circle.*

Solved by Robert Breusch, Michael Goldberg, Marlow Sholander, and (partially) by Sidney Glusman, Ronald Graf, and the proposers.

A Polynomial Assuming Positive Values Only

4781 [1958, 212]. *Proposed by J. L. Massera, Mathematics Institute, Montevideo, Uruguay*

Let $f(x, y, z, \cdots) = a_0(y, z, \cdots)x^n + a_1(y, z, \cdots)x^{n-1} + \cdots + a_n(y, z, \cdots)$, where the a_i are any real functions defined in any region G of

the $(y, z, \cdots)$-space. Let $g(y, z, \cdots)$ be any real function defined in G and construct $f^*(x, y, z, \cdots) = f + gf_x + g^2 f_{xx} + \cdots$. Then, if $f \geqq 0$ in the cylinder $K = G \times \{x: -\infty < x < +\infty\}$, we have $f^* \geqq 0$ in K. More precisely, if the a_i do not vanish simultaneously in G, $f^* > 0$ in K except at points where $f = g = 0$.

Solution by Robert Breusch, Amherst College. There seems to be no significant difference in content between the given theorem and the following:

If $f(x) \equiv a_0 x^n + a_1 x^{n-1} + \cdots + a_n$ (a_i constant, $a_0 \neq 0$), *if* $f(x) \geqq 0$ *for every* x (thus n even, $a_0 > 0$), *if* g *is a constant distinct from zero, then* $f^*(x) \equiv f(x) + g \cdot f'(x) + g^2 \cdot f''(x) + \cdots + g^n \cdot f^{(n)}(x) > 0$ for every x.

Proof. We have

$$f^*(x) = f(x) + g \cdot f^{*\prime}(x) \tag{1}$$

where $f^{*\prime}(x)$ is the derivative of $f^*(x)$. $f^*(x)$ is again a polynomial of even degree, with a positive highest coefficient. Thus $f^*(x)$ is positive for large x. If $f^*(x)$ has a minimum at $x = c$, then $f^{*\prime}(c) = 0$. Thus, from (1), $f^*(c) = f(c) \geqq 0$. Therefore $f^*(x)$ must be nonnegative for every x. In order to show that $f^*(x)$ must be positive (that is, $f(c) \neq 0$), we solve the differential equation (1), obtaining

$$f^*(x) = -\frac{1}{g} e^{x/g} \int_c^x e^{-z/g} f(x)\,dx + f^*(c) \cdot e^{(x-c)/g}.$$

Since the integrand is nonnegative, the integral will be positive or negative, depending on the sign of $x - c$. Thus, if $f^*(c)$ were zero, $f^*(x)$ would be negative for some x, which contradicts the previously-established fact that $f^*(x)$ is nonnegative for all x.

Also solved by the proposer.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

Note. Readers may be interested to know that section 135.215a of the new (August 1958) U. S. Postal Rates provides a special rate of 4 cents for the first pound and 1 cent for each additional pound on books and other library materials, whether printed, photographed, duplicated or typed (including unpublished manuscripts), which are being loaned or exchanged (not sold) *between educational institutions and/or nonprofit organizations or associations.* The wrapper should be marked *LIBRARY MATERIALS* to qualify for this special rate.

Problems in Euclidean Space. Application of Convexity. By H. G. Eggleston. Pergamon Press, New York, 1957. viii+165 pp. $6.50.

This book contains the Adams Prize Essay of the University of Cambridge 1955–6. Professor Eggleston presents the material contained in a series of his own research papers as discussions of ten problems. Problems range from ones concerned with meromorphic functions and planar homeomorphisms to closest packing problems. Mathematicians who proposed some of the problems include: W. Gross, S. M. Ulam, A. S. Besicovitch, C. H. Dowker, Ellen F. and R. C. Buck, and the reviewer. The book is of interest to research workers in geometry, topology, and analysis for its results and for the methods used.

PRESTON C. HAMMER
University of Wisconsin

Introduction to Riemann Surfaces. By George Springer. Addison-Wesley, Reading, Mass., 1957. viii+307 pp. $9.50.

The book deals with abstract Riemann surfaces and their connection with algebraic functions and their integrals. It is not a survey of the subject, but is "a modern presentation of the classical theory which will prepare the reader for further study in this and related fields." It is the first book of its nature written in English, and is certainly a long-awaited text due to the increase of interest in the subject.

The book begins with an Introduction, which bridges the gap between the classical and abstract notions and outlines the goals of the study together with their means. The second chapter, on General Topology, discusses the basic notions to be used for defining an abstract Riemann surface or analytic manifold. It is shown in Chapter 3 that the Riemann surface of an analytic function is also an abstract Riemann surface. Chapter 4 on Covering Manifolds and Chapter 5 on Combinatorical Topology include such topics as covering surfaces, fundamental groups, triangulation, orientability, normal forms of surfaces and homology groups. In Chapter 6, differentials and their integrals on a surface are considered. These differentials form a Hilbert space, which is studied in Chapter 7. The existence of harmonic and analytic differentials and functions on the abstract Riemann surfaces is given in Chapter 8; the problem of finding a uniformizing parameter over the whole surface and automorphic functions are taken up in Chapter 9; and, finally, the meromorphic functions and multiple-valued functions on the Riemann surface are studied in Chapter 10, which connects the study of closed Riemann surfaces and the study of algebraic functions as planned.

The book is written with unusual clearness. As in the Introduction, which outlines the whole book, similar lines appear in each Chapter. There are a list of references, an index, and a collection of exercises at the end of each chapter. The author spends almost one-third of the book presenting a modern treatment in a self-contained manner with a minimum assumption of knowledge—a knowl-

edge of elementary complex function theory and some real variables and algebra. He is most successful in this magnificent project. The book offers not only an excellent treatment of Riemann surfaces, but also an excellent introduction to Topology, Hilbert-space theory and other important mathematical notions. It is highly recommended as a text in its field.

T. K. PAN
University of Oklahoma and
National Taiwan University

Circles. By D. Pedoe. Pergamon Press, New York, 1957. x+78 pp. $3.75.

The prestige of the circle is great. It was deemed to be the only curve fit to be used in geometrical constructions, with the assistance of the straight line (Plato), and even that help may be dispensed with (Mascheroni). Celestial bodies could follow no other path but the perfect one—a circle. Biologists ungraciously point out that nature, in fashioning the organisms of the animal kingdom, has no use for the circle. But Pedoe is still sure that the circle can be used to make friends for—and influence people in favor of—mathematics.

He begins by treating his readers to a few choice bits (or bites?) of modern geometry of the triangle and the circle—and real titbits they are—served in a most attractive way. The elements of the theory of inversion, thrown in for good measure, enable the author, among other things, to exhibit later Poincare's model of Lobacevskian geometry. He shows the advantages that may be derived from the study of the circle by the fact that both the circle in the plane and a point in three-space are determined by three parameters. He even braves the difficulty of trying to convince the reader of the isoperimetric property of the circle.

All this is accomplished with elementary means, in an elegant manner, in the space of a small and slender volume of a few dozen pages. It is to be regretted that, presumably, this economy of space made the author resist the temptation of enhancing the reader's pleasure by including some historical data, or egging on the curiosity of the reader by a few bibliographical references.

NATHAN ALTSHILLER COURT
University of Oklahoma

Elements of Modern Abstract Algebra. By Kenneth S. Miller. Harper, New York, 1958. vii+188 pp. $5.00.

Professor Miller has produced in this text a display of mathematical austerity for "upperclassmen mathematics majors or beginning graduate students." The book is presented in four parts: groups, rings and ideals, fields, sets. It is a compact affair which hurries the reader along from the definition of a group (page 1), through homomorphism theory and the Jordan-Hölder Theorem, to the basis theorem for finite abelian groups (page 49). This pace keeps up throughout the book, and by the time page 184 is reached the reader has been

exposed to ideals, the Hilbert Basis Theorem, finite fields, Zorn's Lemma, and much more. To pursue the subject so relentlessly is not hard if one is willing to keep his eyes on the center of the road and dares only to glance at the countryside. From remarks in the preface (which contains a reference to "ideals in the field of real numbers"), this relentless pursuit seems to be the author's goal; and the reviewer must congratulate the author for standing by his promise. To do so much in so short a space, of course, entails the omission of something—and there isn't *much* choice. In presenting a theorem, one can hardly omit the statement of the theorem; once a theorem is stated, the reader expects shortly to see a proof; and examples are nice to have. That leaves motivation, history, application, *etc.*, as the only possible exclusions and, except for a slight bow now and then to these considerations, they are excluded. The chapters remain unspoiled by problems until the end.

Is this bad? It depends on how the text is used. With so little history and application, there is plenty of room for the *teacher* to ply his trade, and so some might like it. It's quite like "instant algebra" in that something needs to be added. On the other hand, the undergraduate who can read this book and enjoy it without instruction doesn't need to read it—he can go on to Jacobson for bigger and better thrills.

The author has some disconcerting habits. We mention a few.

(1) Every display is an "equation," a typical example being "Equation 1.16": G, G_{16}, G_5, G_1.

(2) Some theorems do not "fit in" as they should. For example, Theorem 12 of Chapter I would be better presented if it preceded Theorem 10. Theorem 4 of Chapter II implies Theorem 3.

(3) Examples are not always instructive: one example of a ring without identity is lost to the reader who is not familiar with the Dirac δ-function.

(4) Some results are never subsequently used, so that the reader can only wonder about their inclusion.

The book contains relatively few errors. Helpful "identifications" are made in cases where some authors persist in pedantry, and there are but a few theorems or proofs which are unnecessarily complicated. The section on sets (an appendix) is nice to have, and applications are given.

ROBERT J. WISNER
Haverford College

Commutative Algebra, Vol. I. By Oscar Zariski and Pierre Samuel. University Series in Higher Mathematics, Van Nostrand, Princeton, 1958. xi+330 pp. $6.95.

The authors approach modern algebra in this book from a sophisticated viewpoint. The theory of the structure of rings is developed in as general a way as feasible except for the assumption of commutativity. Since this book is intended as a companion to the second volume, to be published on algebraic geometry, the restriction is natural. Despite its dual purpose, the book is an excellent

reference book on commutative algebra.

As a reference book, it contains no problems; the function of illustrating the theory is adequately served by copious "remarks" and "examples." An occasional minute detailing of a proof is an interesting exercise but has little place in a reference book. Luckily, such tendencies are restricted to the first chapter.

Careful indexing of the extensive collection of definitions makes them readily available for reference. The definitions are introduced in the text as they are used; some are even dragged in where they might be omitted in order to make the list more complete.

The first three chapters contain "basic definitions and properties of algebraic structures." Chapter I introduces most of the basic systems: groups, rings, fields, polynomial rings, and vector spaces. Chapter II develops the theory of extension fields. Chapter III is concerned with ideals and modules and the structure of rings developable from them.

The last two chapters restrict the basic ring and develop further the structure theory of the specialized rings. Chapter IV is primarily restricted to Noetherian rings while Chapter V develops the theory of integral dependence and of Dedekind domains.

The writing is uneven in quality and at times suffers from the dry, frequently dull style so common in mathematical papers; but it has the overwhelming compensation of being almost always clear. In part the authors and in part the publisher are responsible for many small errors ranging from broken letters in the type to questionable or even incorrect statements. A random count on twenty pages yielded at least eight such flaws, none of which seriously affected the readability of the pages. Also, the format of the book is variable. In particular, Chapter V is quite different in form from the previous ones.

The book is clearly written, presenting in a sophisticated comprehensive way a connected body of material, much of which was not previously available in book form. The index of definitions alone would be sufficient to persuade this reviewer to recommend to every mathematician that he include *Commutative Algebra* in his reference library.

DONALD A. NORTON
University of California, Davis

BRIEF MENTION

Advances in Applied Mechanics, Vol. 5. Edited by H. L. Dryden, Th. von Karman, and G. Kuerti. Academic Press, New York, 1958. x+459 pp. $12.00.

The fifth volume of *Advances in Applied Mechanics* contains surveys and reviews of research in applied mechanics. The volume is divided into seven portions. These are: Supersonic Air Ejectors, Unsteady Airfoil Theory, The Theory of Distributions, Stress Wave Propagation in Rods and Beams, Problems in Hydromagnetics, Mechanics of Granular Matter, and Condensation in Super-

sonic and Hypersonic Wind Tunnels. The individual papers will undoubtedly be reviewed in appropriate journals. We call this volume to our readers' attention.

Algebraic Geometry and Topology. A symposium in honor of S. Lefschetz. Ed. by R. H. Fox, D. C. Spencer, and A. W. Tucker. Princeton University Press, New Jersey, 1957. viii+399 pp. $7.50.

What more appropriate birthday present could there be for a great mathematician than a collection of contemporary research papers which have developed from his own basic work? The individual papers in this volume will undoubtedly be reviewed in appropriate journals; indeed, several of them have already been so reviewed. We commend this collection to readers interested in mathematics in general as well as those interested in algebraic geometry and topology.

Algebra. By W. L. Ferrar. Oxford University Press, New York, 1958. vii+220 pp. $2.80.

Professor Ferrar has added a chapter on latent vectors to his 1941 textbook on determinants, matrices, and algebraic forms.

Analytic Geometry of Three Dimensions. By George Salmon. Seventh Edition. Chelsea, New York, 1958. xxiv+470 pp. $4.95.

The printing of this volume, which was already in its fourth edition in 1882, later revised by Reginald A. P. Rogers in 1911, and the revision re-edited into the (current) seventh edition by C. H. Rowe in 1927 probably sets some sort of record for longevity in 19th and 20th century books.

Calculus of Finite Differences. By George Boole. Fourth Edition. Chelsea, New York, 1958. xii+336 pp. $4.95.

Another old-timer, Boole's 1860 treatise, revised somewhat by John F. Moulton in 1872.

College Mathematics. By Kaj L. Nielsen. Barnes and Noble College Outline Series, New York, 1958. xviii+302 pp. $1.95.

This outline, keyed to the general course intended for students who do not plan to continue the study of mathematics, contains fifteen pages of sample examination questions along with answers thereto.

Modern Computing Methods. N. P. L. Staff. Philosophical Library, New York, 1958. vi+129 pp. $8.75.

These notes are based on lectures from a course on "Computers for Electrical Engineering Problems" presented to representatives of industrial firms. Perhaps the best feature of the volume is the twelve-page annotated bibliography (before 1956) which is appended.

Notes on Analog-Digital Conversion Techniques. Edited by Alfred K. Susskind. Wiley, New York, 1957. x+410 pp. $10.00.

These notes, the results of intensive one-week courses presented in the summers of 1956 and 1957 at Massachusetts Institute of Technology, present the subject matter for a practicing engineer having only moderate sophistication. In addition to an introductory chapter, which discusses systems in general, there are chapters on Sampling and Quantizing, Codes, Digital Circuits, Coding and Decoding Techniques for Electrical Signals, Coding and Decoding Techniques for Translational and Angular Motion, Design of a Digital Instrumentation System, and Optical Coders for DFTI.

Basic Geometry. By George D. Birkhoff and Ralph Beatley. Chelsea, New York, 1958. 294 pp. $3.95.

It seems most appropriate that this book, first published by the authors in an experimental edition in 1933 and later published by Scott, Foresman and Company in 1940–41, be reprinted at this time. Certainly anyone interested in teaching geometry or in writing a text on geometry could well spend a few hours studying this book.

NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to L. J. Montzingo, Jr., Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

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Please do not throw away this issue. If you do not regularly file or pass your copies on after reading them, why not participate in the Magazines for Friendship Program? This nonprofit organization will provide you with selected names of foreign scholars, teachers, universities and libraries eager to receive learned U. S. publications, even old ones. For complete details, please send a stamped, self-addressed envelope to Magazines for Friendship, Occidental College, Los Angeles 41, California.

VISITING ASSOCIATESHIPS IN TEST DEVELOPMENT

Two Visiting Associateships in Test Development are being offered to secondary school or college teachers by the Educational Testing Service, one in Science and one in Mathematics. The appointments will be for July and August, 1959. The Associates will work primarily on tests at the college-entrance and higher levels. They will analyze existing tests and work on planning new ones. The stipend is $700 plus transportation to and from Princeton. Application forms must be submitted by February 27, 1959. All inquiries should be addressed to Mrs. W. Stanley Brown, Test Development Division, Educational Testing Service, 20 Nassau Street, Princeton, New Jersey.

NSF SUMMER INSTITUTES FOR MATHEMATICS AND STATISTICS

The National Science Foundation has announced the following institutes in the summer of 1959 for college and high school teachers of mathematics. Inquiries about a particular institute should be sent to the Director named for that institute. Unless otherwise indicated, the Director is located at the same institution as the institute. The group of teachers for whom an institute is intended is given following the dates according to the following code: C—College, JC—Junior College, HS—High School, JHS—Junior High School, SHS—Senior High School.

Mathematics

Arizona State University, Tempe, June 22–July 31: HS. Lloyd L. Lowenstein.

University of Arizona, Tucson, June 8–July 18: JHS. Millard G. Seeley.

Loyola University, Los Angeles, Calif., June 22–July 31: J&SHS. B. R. Wicker.

San Jose State College, San Jose, Calif., June 22–July 31: HS. Max Kramer.

University of Santa Clara, Santa Clara, Calif., June 22–August 1: HS. Irving Sussman.

University of California, Berkeley, June 22–July 31: J&SHS&C. Frantisek Wolf.

University of California, Los Angeles, July 6–August 28: C. C. B. Tompkins.

University of Southern California, Los Angeles, June 22–July 31: C. D. Victor Steed.

Catholic University of America, Washington, D. C., June 29–August 7: SHS. Raymond W. Moller.

Georgetown University, Washington, D. C., July 6–August 14: SHS. Malcolm W. Oliphant.

Eastern Illinois University, Charleston, Ill., June 15–August 7: JHS-HS. Lawrence A. Ringenberg.

Knox College, Galesburg, Ill., July 6–August 14: HS. Rothwell Stephens.

Northwestern University, Evanston, Ill., June 22–August 15: J&SHS. E. H. C. Hildebrandt.

Ball State Teachers College, Muncie, Ind., July 20–August 21: HS-C. Charles F. Brumfiel.

Indiana University, Bloomington, June 29–August 7: SHS. Mrs. Marie S. Wilcox, Thomas Carr Howe High School, Indianapolis, Ind.

University of Notre Dame, Notre Dame, Ind., June 19–August 4: J&SHS. Arnold E. Ross.

Purdue University, Lafayette, Ind., June 8–July 31: SHS. M. Wiles Keller.

Purdue University, Lafayette, Ind., June 8–July 31: J&SHS. G. N. Wollan.

Drake University, Des Moines, Iowa, June 8–July 17: J&SHS. Basil E. Gillam.

Iowa State College, Ames, June 8–July 18: C. J. A. Greenlee.

State University of Iowa, Iowa City, June 22–August 2: SHS. Lloyd A. Knowler.

Fort Hays Kansas State College, Hays, Kan., June 3–July 30: J&SHS. W. Toalson.

Kansas State College, Manhattan, June 15–August 7: HS. Leonard E. Fuller.

University of Kansas, Lawrence, June 8–August 1: SHS-C. G. Baley Price.

Washburn University of Topeka, Topeka, Kan., June 10–August 5: JHS. Laura Z. Greene.

Bowdoin College, Brunswick, Me., June 29–August 8: SHS. Dan E. Christie.

University of Maine, Orono, July 6–August 14: SHS. S. H. Kimball.

University of Maryland, College Park, June 22–July 31: JHS. John Brace.

Boston College, Chestnut Hill, Mass., July 6–August 14: J&SHS. Rev. Stanley J. Bezuszka, S.J.

Clark University, Worcester, Mass., June 28–August 14: SHS&JC. Charles T. Bumer.

College of the Holy Cross, Worcester, Mass., June 29–August 7: J&SHS. Rev. Raymond J. Swords, S.J.

University of Massachusetts, Amherst, June 29–August 14: J&SHS. Robert W. Wagner.

Central Michigan College, Mt. Pleasant, Mich., June 22–July 31: JHS-HS. Lester H. Serier.

Eastern Michigan College, Ypsilanti, Mich., June 22–July 31: SHS. Robert S. Pate.

University of Michigan, Ann Arbor, June 29–August 8: SHS. Bernard A. Galler.

Wayne State University, Detroit, Mich., June 22–August 14: J&SHS. Karl W. Folley.

Western Michigan University, Kalamazoo, Mich., June 22–July 31: SHS. Charles H. Butler.

St. Louis University, St. Louis, Mo., June 15–July 24: JHS-HS. Francis Regan.

Southwest Missouri State College, Springfield, Mo., June 15–July 24: JHS. Carl V. Fronabarger.

Montana State College, Bozeman, July 20–August 21: J&SHS. Adrien L. Hess.

Montana State University, Missoula, June 29–August 7: C. Frederick H. Young.

Princeton University, Princeton, N. J., June 29–August 7: HS&C. J. A. Farrington, Jr.

Rutgers, the State University, New Brunswick, N. J., June 29–August 7: J&SHS. Emory P. Starke.

Montclair State College, Upper Montclair, N. J., June 29–August 7: J&SHS. Max A. Sobel.

University of New Mexico, Albuquerque, June 20–August 14: J&SHS. Frank C. Gentry.

Brooklyn College, Brooklyn, N. Y., July 6–August 7: J&SHS. Carroll W. Grant.

University of Buffalo, Buffalo, N. Y., July 6–31: HS. Harriet F. Montague.

Teachers College, Columbia University, New York, July 6–August 14: SHS. Howard F. Fehr.

Hamilton College, Clinton, N. Y., June 29–August 22: J&SHS. Brewster H. Gere.

Hunter College, New York, July 1–August 8: J&SHS. Jewell Hughes Bushey.

New York State College for Teachers, Albany, N. Y., June 29–August 8: J&SHS. Edgar W. Flinton.

University of Rochester, Rochester, N. Y., June 29–August 8: SHS. William A. Fullagar.

Syracuse University, Syracuse, N. Y., June 29–August 8: J&SHS. Robert B. Davis.

Duke University, Durham, N. C., June 15–July 24: C. J. J. Gergen.

Baldwin-Wallace College, Berea, Ohio, June 22–July 31: SHS. Dean L. Robb.

Case Institute of Technology, Cleveland, Ohio, June 21–July 31: J&SHS. Paul E. Guenther.

University of Cincinnati, Cincinnati, Ohio, June 26–August 7: SHS. H. David Lipsich.

Kent State University, Kent, Ohio, June 22–August 14: SHS. Kenneth B. Cummins.

Oberlin College, Oberlin, Ohio, June 15–August 7: SHS. Wade Ellis.

Oklahoma State University, Stillwater, June 15–July 25: J&SHS-C. James H. Zant.

Oregon State College, Corvallis, June 22–August 14: J&SHS. Albert R. Poole.

University of Oregon, Eugene, June 22–August 15: J&SHS. A. F. Moursund.

Portland State College, Portland, Ore., July 13–August 28: HS. Robert W. Rempfer.

Reed College, Portland, Ore., June 20–August 14: J&SHS-JC. Burrowes Hunt.

Lehigh University, Bethlehem, Pa., June 22–August 1: SHS. Clarence A. Shook.

University of Pittsburgh, Pittsburgh, Pa., June 22–August 14: HS. John C. Knipp.

Seton Hill College, Greensburg, Pa., June 29–August 7: J&SHS. Sister Mary Thaddeus.

Catholic University of Puerto Rico, June 22–July 31: J&SHS. K. C. Schraut, University of Dayton, Dayton 9, Ohio.

University of Puerto Rico, Mayagüez, Puerto Rico, June 8–July 17: J&SHS. Mariano García.

University of South Carolina, Columbia, June 15–August 10: HS-C. W. L. Williams.

Memphis State University, Memphis, Tenn., June 8–July 10: SHS. H. S. Kaltenborn.

Vanderbilt University, Nashville, Tenn., June 8–July 31: J&SHS. E. Baylis Shanks.

East Texas State College, Commerce, Tex., June 3–July 14: SHS. Roy N. Jervis.

Our Lady of the Lake College, San Antonio, Tex., June 3–July 16: H.S. Sister M. Laetitia Hill.

Southern Methodist University, Dallas, Tex., July 15–August 25: J&SHS. Joe P. Harris, Jr.

University of Vermont, Burlington, June 29–August 14: SHS. N. James Shoonmaker.

State College of Washington, Pullman, June 15–August 7: SHS. Sidney G. Hacker.

Western Washington College of Education, Bellingham, Wash., June 22–August 21: H.S. Harvey M. Gelder.

University of Wyoming, Laramie, June 15–August 7: C. W. Norman Smith.

Statistics

North Carolina State College, Raleigh, June 8–July 17: C. F. E. McVay.

Oklahoma State University, Stillwater, June 8–July 31: C. Carl E. Marshall.

University of Wyoming, Laramie, June 15–August 7: C. Edward C. Bryant.

NATIONAL REGISTER OF SCIENTIFIC AND TECHNICAL PERSONNEL

On behalf of the Mathematical Association of America and various other mathematical organizations, the American Mathematical Society is assembling and maintaining a register of mathematicians and mathematical scientists. The mathematics register is a section of the National Register of Scientific and Technical Personnel, which is an official responsibility of the National Science Foundation. The purpose of the Register is to provide up-to-date information on the scientific manpower resources of the United States.

As a result of the splendid cooperation accorded to the project by most of the mathematicians and mathematical scientists who have received questionnaires to fill in, the mathematical section of the Register is now remarkably complete. However, there are still a few gaps to be filled in.

If you have received a National Register questionnaire from the American Mathematical Society, won't you please fill it in now and send it to the Headquarters Offices of the Society at 190 Hope Street, Providence 6, Rhode Island?

If you have never received a questionnaire and feel that you are qualified for inclusion in the Register, please drop us a note to that effect at the above address.

PERSONAL ITEMS

Professor L. W. Cohen of the University of Maryland was the representative of the Association at the annual meeting of the American Council on Education held in Chicago on October 9 and 10, 1958.

Dean Mina Rees of Hunter College represented the Association at the Inauguration of President H. W. Stoke of Queens College, New York, on Wednesday, October 22, 1958.

Brigham Young University: Associate Professor H. J. Fletcher has been appointed Chairman of the Mathematics Department; Assistant Professor Donald Robinson has been promoted to Associate Professor.

The Florida State University: Associate Professor E. P. Miles, Jr., Alabama Polytechnic Institute, has been appointed Associate Professor; Dr. John Greever, University of Virginia, Dr. Gabriel Margulies, Indiana University, Dr. L. L. Lasman, North Carolina State College, and Dr. M. F. Tinsley, Ohio State University, have been appointed Assistant Professors; Mr. C. A. Brown and Mr. B. L. Sanders have been appointed Instructors.

Georgetown University: Dr. F. G. Asenjo, University De La Plata, Argentina, Dr. J. E. LeBel, University of Toronto, and Mr. J. I. Hincke, United States Army, have been appointed to the Mathematics Faculty.

University of Georgia: Professor M. K. Fort, Jr. has been awarded an Alfred P. Sloan Fellowship for 1958–59 and 1959–60; Associate Professor M. L. Curtis has been promoted to Professor; Dr. R. P. Hunter and Dr. J. J. Andrews have been appointed Assistant Professors.

Rutgers, The State University: Assistant Professors Solomon Leader and K. C. Wolfson have been promoted to Associate Professors; Mr. W. R. Jones has been promoted to Instructor; Dr. J. M. Danskin, Jr., Institute for Advanced Study, has been appointed Assistant Professor; Mr. Terence Butler, Indiana University, Mr. M. J. Greenberg, Princeton University, Dr. P. E. Martin, Harvard University, and Mr. Stephen Weingram, Princeton University, have been appointed Instructors.

State College of Washington: Dr. Helmut Schaefer of Mainz, Germany, has been appointed Associate Professor; Dr. T. R. Jenkins, Lockheed Aircraft Corporation, and Assistant Professor T. A. Newton, Colorado State University, have been appointed Assistant Professors; Dr. Tanjiro Okubo of the National Defense Academy, Yokosuda, Japan, has been appointed Visiting Assistant Professor for the present academic year.

University of Texas: Associate Professor W. T. Guy, Jr. has been appointed Acting Chairman of the Department of Mathematics; Assistant Professor F. N. Edmonds, has been promoted to Associate Professor and is on a research leave for the academic year 1958–59; Assistant Professor E. J. Prouse has been promoted to Associate Professor; Dr. Ben Fitzpatrick, Jr. has been promoted to Assistant Professor; Dr. D. M. Young, Ramo-Woolridge, Los Angeles, has been appointed Professor and Director of the Computing Center.

Mathematics Research Center, United States Army: Dr. R. N. Buchal, New York University, Dr. H. F. Bueckner, General Electric Company, Schenectady, New York, and Dr. A. Ghaffari, National Bureau of Standards, Washington, D. C., have joined the staff; Dr. Miklos Hetenyi, on leave from Northwestern University, began an appointment at the Center and also holds a Visiting Professorship in Mechanics in the College of Engineering at the University of Wisconsin; Dr. J. A. Nohel, on leave from Georgia Institute of Technology, and Dr. A. L. Rabenstein, Massachusetts Institute of Technology, will spend a year at the Center; Dr. C. H. Wilcox, on leave from California Institute of Technology, and Dr. R. G. D. Steel, on leave from Cornell University, hold visiting appointments.

Mr. John Abramowich, University of California, Berkeley, has been appointed Instructor at the University of British Columbia, Vancouver.

Mr. R. J. Andree, Oklahoma State University, has accepted a position as Mathematical Analyst with Lockheed Missile Systems Division, Palo Alto, California.

Associate Professor I. L. Battin, Drew University, has been appointed Professor at New Jersey State Teachers College, Trenton.

Mr. Ralph Beals, University of Kentucky, is now a Teaching Assistant at Northwestern University.

Professor R. F. Bell, on leave from Eastern Washington College of Education, is a Lecturer at the University of Michigan.

Associate Professor Barney Bissinger, Lebanon Valley College, is spending the year on leave with the Statistical Research Group at Princeton University under a National Science Foundation Science Faculty Fellowship.

Mr. T. H. Blackburn, Case Institute of Technology, has been appointed Professor and Chairman of the Department of Mathematics at Lenoir Rhyne College.

Mr. K. Z. Bradford, University of Oklahoma, has accepted a position as Physicist with Hughes Aircraft Company, Culver City, California.

Professor J. A. Brown, State University of New York, Teachers College at Oneonta, has been appointed Associate Professor at the University of Delaware.

Mr. J. R. Brown, Oregon State College, is a University Fellow at Yale University Graduate School.

Mr. C. M. Bruen, I.B.M. Corporation, has been promoted to Associate Mathematician with I.B.M. Scientific Computation Laboratory, Endicott, New York.

Professor Herbert Busemann, on leave from the University of Southern California, is a Visiting Professor at Harvard University.

Mr. B. R. Buzby, Indiana University, has accepted a position as Research Mathematician with Metals Research, Electro Metallurgical Company, Niagara Falls, New York.

Mr. A. J. Carlan, Hoffman Semiconductor Division, Evanston, Illinois, is now a Fellow at the Mellon Institute.

Mr. R. K. Clark, Texas Christian University, has been appointed Teaching Assistant at the University of California, Berkeley.

Mr. W. L. Congleton, Bell Telephone Laboratories, Murray Hill, New Jersey, has accepted the position of Electrical Engineer with Sylvania Electrical Products, Needham, Massachusetts.

Dr. S. D. Conte has been promoted to Manager of the Mathematical Analysis Department of the Space Technology Laboratories, Los Angeles, California.

Associate Professor Byron Cosby, Jr., State University of Iowa, has been appointed Professor of Actuarial Science at the University of Texas.

Associate Professor C. H. Cunkle, Dickinson College, has accepted a position as Research Mathematician with Cornell Aeronautical Laboratory, Inc., Buffalo, New York.

Mr. Charles Drescher, Western Electric Company, New York City, is now a Member of the Technical Staff of the Ramo-Wooldridge Corporation, Los Angeles, California.

Miss Ann Farek, Laredo Junior College, has been appointed Instructor at the Texas College of Arts and Industries.

Mr. W. E. Felling, Parks College of Aeronautical Technology, has accepted a position as Research Scientist with the McDonnell Aircraft Corporation, St. Louis, Missouri.

Mr. W. E. Ferguson, on leave from Newton High School, Newtonville, Massachusetts, is a Visiting Lecturer at the University of Illinois, 1958–59.

Mr. H. E. Fleming, Harpur College, is now a Teaching Assistant at the University of Maryland.

Dr. Abraham Franck, Senior Mathematician with Engineering Research Associates of Remington Rand, St. Paul, Minnesota, has been promoted to Manager, Systems and Mathematics Research.

Mr. D. A. Freedman, McGill University, Montreal, is now a Rand Fellow in Mathematical Statistics, Princeton University.

Mr. J. D. Gilbert, Alabama Polytechnic Institute, has been appointed Assistant Professor at Louisiana Polytechnic Institute.

Associate Professor W. M. Gilbert, on leave from Iowa State College, is a Visiting Fellow at Princeton University.

Associate Professor E. H. Gilmore, Oklahoma State University, has been appointed Assistant Professor at Texas Technological College.

Mr. M. L. Glasser, University of Miami, has been appointed Research Assistant at Carnegie Institute of Technology.

Mr. J. B. Goebel, University of Oregon, has been appointed Teaching Fellow at Oregon State College.

Professor M. O. Gonzalez, University of Havana, has been appointed Professor at the University of Alabama.

Mr. William Granet, Office of Statistical and Research Services, Boston University, has been appointed Director of the Computing Center, Oklahoma State University.

Mr. J. R. Hanne, Dartmouth College, is now a Teaching Fellow at the University of Michigan.

Dr. S. M. Harmon, University of California, Los Angeles, has been appointed Assistant Professor at Fresno State College.

Mr. F. H. Hildebrand, Kent State University, is an NSF Fellow at the University of Illinois for the academic year 1958–59.

Mr. Raymond Hirschkop, Pratt Institute, is now on the Research Staff at Lincoln Laboratories, Massachusetts Institute of Technology.

Assistant Professor T. C. Holyoke, Miami University, Ohio, has been appointed Associate Professor at Antioch College.

Mr. Norman Johnson, Geneva College, is now a Teaching Fellow at the University of Toronto.

Assistant Professor J. B. Johnston, University of Kansas City, has been appointed Assistant Professor at the University of Kansas.

Mr. A. P. Jones, U. S. Army Computing Laboratory, is now a Mathematician with the National Heart Institute, Bethesda, Maryland.

Assistant Professor R. P. Kelisky, University of Texas, has accepted the position of Associate Mathematician with I.B.M. Research Center, Yorktown Heights, New York.

Dr. Naoki Kimura, Tokyo Institute of Technology, Japan, has been appointed Acting Assistant Professor at the University of Washington.

Mr. G. D. King, Brevard College, has been appointed Assistant Professor at Clemson College.

Associate Professor O. M. Klose, Seattle University, has been appointed Associate Professor at Humboldt State College.

Dr. G. R. Lehner, University of Wisconsin, has been appointed Instructor at the University of Maryland.

Mr. R. J. Libera, University of Massachusetts, has been appointed Teaching Assistant at Rutgers, The State University.

Associate Professor A. C. Lindberg, Dana College, has been appointed Assistant Professor at Mankato State College.

Mr. W. M. Lowney, University of Notre Dame, has accepted the position of Scientist with Lockheed Missiles System Division, Palo Alto, Califfornia.

Mr. J. C. Mairhuber, University of Rochester, has been appointed Assistant Professor at the University of New Hampshire.

Mr. Dale Maness, Chance Vought Aircraft Inc., Dallas, Texas, has been appointed Professor and Chairman of the Department of Mathematics at Howard Payne College.

Mr. H. T. Mathews, Georgia Institute of Technology, has been appointed Associate in Mathematics at Louisiana State University in New Orleans.

Dr. Elliott Mendelson, Harvard University, has been appointed Instructor at Columbia University.

Dr. Mabel D. Montgomery, University of Buffalo, has been appointed Associate Professor at State University of New York, College for Teachers at Buffalo.

Dr. H. S. Moredock, Jr. has been appointed Professor at Sacramento State College.

Mr. D. L. Muench, Weapon Systems Laboratory, Aberdeen Proving Ground, Maryland, has been appointed Graduate Assistant at St. John's University, New York.

Mr. M. J. Pascual has been appointed Assistant Professor at Siena College.

Dr. M. J. Poliferno, Williams College, has been appointed Instructor at Trinity College, Connecticut.

Dr. G. Y. Rainich, on leave from the University of Michigan, is a Visiting Professor at the University of Notre Dame.

Mr. C. R. Riehm, University of Toronto, has been appointed Assistant in Research at Princeton University.

Mr. E. L. Roetman, Indiana University, has been appointed Teaching Assistant at the University of Minnesota.

Mr. H. D. Ruderman, Bronx High School of Science, New York, has been appointed Chairman of the Mathematics Department of Hunter College High School.

Mr. P. T. Rygg has been appointed Instructor at Iowa State College.

Mr. R. T. Sandberg, University of Buffalo, has been appointed Instructor at the University of Arizona.

Dr. B. D. Seckler, Brooklyn College, has been appointed Assistant Professor at Pratt Institute of Technology.

Mr. Anthony Sepan, Temple University, has been appointed Teaching Assistant at the University of Wisconsin.

Mr. W. T. Sharp, on leave from Atomic Energy of Canada Ltd., is an Instructor at Princeton University for the academic year 1958–59.

Dr. M. W. Shelly II, Laboratory of Aviation Psychology, Columbus, Ohio, has been appointed Psychologist at the Office of Naval Research, Deerfield, Illinois.

Mr. M. G. Shults, Northern Oklahoma Junior College, has been appointed Assistant Professor at Panhandle Agricultural and Mechanical College.

Dr. H. F. Simmons, Iowa State College, has been appointed Assistant Professor at California State Polytechnic College.

Mr. R. H. Sprague, University of Kentucky, has been appointed Assistant Professor at New Mexico Agricultural and Mechanical College.

Mr. H. R. Stevens, University of Buffalo, has been appointed Part-Time Instructor at Duke University.

Dr. J. R. Stock, Union Carbide and Carbon Corporation, New York, has been appointed Engineer at the Stock Equipment Company, Cleveland, Ohio.

Mrs. Doris S. Stockton, Brown University, has been appointed Assistant Professor at the University of Massachusetts.

Dr. Karl Stromberg, University of Washington, is now Postdoctoral Research Associate at Yale University.

Mr. L. R. Tappan, University of Michigan, has been appointed Assistant Professor at Nicholls State College.

Mr. J. D. Thomas, University of Oklahoma, has accepted a position as Staff Member at Los Alamos Scientific Laboratory.

Mr. C. C. Thompson, Oklahoma State University, has been appointed Assistant Instructor at Yale University.

Dr. R. N. Townsend, University of Illinois, has been appointed Assistant Professor at San Jose State College.

Dr. R. N. Walter, Manhattan High School of Aviation Trades, New York, has been appointed Professor at Paterson State College.

Professor S. E. Warschawski, on leave from the University of Minnesota, has been appointed Visiting Professor at the University of California at Los Angeles for the academic year 1958–59.

Mr. D. B. Wells, University of Kentucky, has been appointed Instructor at Western Carolina College.

Dr. G. M. Wing, Los Alamos Scientific Laboratory, has been appointed Associate Professor at the University of New Mexico.

Assistant Professor Mildred M. Sullivan, Queens College, New York, died on August 30, 1958. She was a member of the Association for eleven years.

Professor G. C. Vedova, Newark College of Engineering, died on September 5, 1958. He was a member of the Association for twenty years.

Professor Emeritus A. H. Wilson, Haverford College, died on September 22, 1958 at the age of 86 years. He was a charter member of the Association.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 204 persons have been elected to membership by the Board of Governors on applications duly certified.

JEAN E. ABELL, A.A. (California, Davis) Student, University of California, Berkeley.

JACK A. ADJAMI, Student, Brooklyn College.

ARNOLD C. AHLIN, B.A. (Eastern Washington) Mathematician, Lincoln Lab.

ETHAN O. ALLEN, M.A. (Cornell) Chief, Math. Analysis Section, Bell Aircraft Corp.

JOE H. ALLEN, B.S. (Middle Tennessee S.C.) Teacher, Donelson High School, Tennessee.

BROTHER ANDREW, B.S. (Fordham) Head of Dept., St. Anthony's High School, Smithtown, New York.

JERRY G. BAILS, M.A. (Kansas City) Instr., University of Kansas.
CAMERON C. BARR, JR., B.S. (Roanoke) Instr., Roanoke College.
LT. COL. LASZLO BERES, Asst. Military and Air Attache, Hungarian Legation, Washington, D. C.
BERNARD BERGER, Student, University of Pittsburgh.
MARTIN F. BERMAN, B.A. (U.C.L.A.) Asst. Mathematician, System Development Corp.
JOSEPH R. BIENAS, Student, Indiana Technical College.
BERNARD E. BJORK, B.S. (St. Cloud S.C.) Instr., Falls High School, International Falls, Minnesota.
MARTIN BLUMBERG, M.S. in E.E. (Stanford) Eng. Specialist, Electronic Defense Lab., Mountain View, California.
DAVID B. BOOTHBY, M.A. (Massachusetts) Instr., University of Vermont.
MARK BRIDGER, Student, Bronx High School of Science, New York.
JAMES D. BRISTOL, M.A. (Western Reserve) Teacher, Shaker Heights Senior High School, Ohio.
CARL N. BROOKS, Student, University of Maine.
RICHARD A. BROWN, M.S. (S.U. of Iowa) Asso. Professor, Bluefield State College.
ROBERT F. BROWN, A.B. (Harvard) Mathematician, Bureau of Supplies and Accounts, U. S. Navy.
O. LEXTON BUCHANAN, JR., M.Ed. (Georgia) Asst., University of Kansas.
J. BERNARD BUCKY, B.S. (Queens, New York) Grad. Student, New York University.
MRS. JOYCE M. BURCHENAL, B.S. (Michigan Coll. of Mining & Tech.) Head of Dept., L'Anse Creuse High School, Mt. Clemens, Michigan.
NEWBERN W. BUSH, M.S. (Auburn) Head of Dept., Jacksonville State College.
MAGDELHAYNE F. BUTEAU, M.S. (Montreal) Professor, St. Joseph Teachers College.
RICHARD E. CADY, M.S. (West Virginia) Instr., West Virginia University.
HUGH G. CAMPBELL, M.S. (Florida S.U.) Asso. Professor, Virginia Polytechnic Institute.
JAY O. CASEY, Student, Oklahoma State University.
BERNARD C. CHALOUPKA, B.S. (Creighton) Engr., Martin Co.
GEORGE L. CHANEY, M.S. (Kansas S.T.C.) Instr., Coffeyville College.
RICHARD M. CHESSER, Student, Central State College, Oklahoma.
JOSEPH B. CHICCARELLI, M.A. (Boston Coll.) Asst. Professor, Fordham University.
KEEWHAN CHOI, B.S. (Case I.T.) Grad. Student, Massachusetts Institute of Technology.
ALICE M. CHRISTIANSEN, M.S. (Northwestern) Instr., University of Illinois.
MARSHALL B. COHEN, B.S. (Wisconsin) Mathematician, Cornell Aeronautical Lab.
MARSHALL M. COHEN, Student, University of Chicago.
HARVEY COHN, Ph.D. (Harvard) Head of Dept., University of Arizona.
ALEX C. COMPTON, Student, Columbia University.
LEWIS H. COON, M.S. (Indiana) Asst. Professor, Southwestern State College.
DENSEL G. CORBIN, B.S. (Oklahoma) Res. Engr., Jersey Production Research Center.
PATRICIA A. COUGER, M.A. (Wichita) Instr., Wichita University.
WILLIAM D. CRAVEN, B.S. (Pennsylvania S.U.) Grad. Student, Pennsylvania State University
ARTHUR CRAWSHAW, Engr., Washington State Highway Dept., Seattle, Washington.
STERLING C. CRIM, M.A. (Peabody) Asst. Professor, West Georgia College.
JOHN P. CUELLAR, JR., B.S. (St. Mary's, Texas) Junior Res. Engr., Southwest Research Institute.
ETHA A. DAHLGREN, B.A. (Coll. of Mt. St. Vincent) Asst. Supervisor, New York City Dept. of Welfare.
BENJAMIN DAMSKY, B.S. (Glassboro S.T.C.) Teacher, Roosevelt Junior High School, New Brunswick, New Jersey.
STEPHEN P. DILIBERTO, Ph.D. (Princeton) Asso. Professor, University of California, Berkeley.
RUSSELL D. F. DINEEN, M.A. (Delaware) Teacher, Board of Public Schools, Wilmington, Delaware
LESLIE A. DWIGHT, Ph.D. (Geo. Peabody) Head of Dept., Southeastern State College.
BENJAMIN F. EDWARDS, JR., B.S. (S.F. Austin S.C.) Senior Engr., Chance Vought Aircraft Inc.
VALMA Y. EDWARDS, B.S. (Florida S.U.) Grad. Asst., Florida State University.
MRS. FLORENCE L. ELDER, M.A. (Columbia) Chairman of Dept., W. Hempstead Junior-Senior High School, New York.
EDWIN L. ELLIS, Ballistic Research Lab., Aberdeen Proving Ground, Maryland.
JOHN C. ESTY, JR., M.A. (Yale) Asso. Dean and Instr., Amherst College.
CLEMENT E. FALBO, M.A. (Texas) Instr., San Antonio College.
LT. NEAL A. FARMER, B.S. (Tusculum Coll.) United States Marine Corps.
JOHN D. FERRUCCI, M.S. (Kansas S.C.) Instr., Teachers College of Connecticut.
WILLIAM B. FLOYD, M.S. (Emory) Senior Res. Engr., Melpar Inc..
JOHN A. FLYNN, Engr., Martin-Denver.
JOHN C. FOGARTY, Student, Harvard University.
DONALD FORBES, Asso. Editor and Senior Economist, Econometric Institute.
STANLEY P. FRANKLIN, Student, Memphis State University.
RICHARD B. FREY, M.S. (Illinois) High School Instr., Des Plaines, Illinois.
HERMAN P. FRIEDMAN, M.A. (Brooklyn) Senior Mathematician, Bulova Research and Development Labs.
KURT O. FRIEDRICHS, Ph.D. (Göttingen) Professor, New York University; Asso. Director, Institute of Mathematical Sciences.
PHILLIP H. C. FUNG, Student, Idaho State College.
JOHN W. GAMMILL, B.S. (Delta S.C.) Instr., Mississippi State College.
CHARLES E. GARDNER, JR., Student, Oklahoma State University.
JOHN B. GARNER, Res. Asst., Carleton College.
JOHN M. GARY, Ph.D. (Michigan) Instr., California Institute of Technology.
ROBERT M. GASPER, Student, Butler University.
JOHN H. GAY, Ph.D. (Columbia) Dean of Instruction, Cuttington College, Monrovia, Liberia.
JOSEPH B. GEISER, M.S. (Chicago) Teacher, University of Chicago Lab. School.
CALVIN W. GILLARD, B.S. (U.C.L.A.) Engr., Sylvania Electric Products Inc.; Grad. Student, Stanford University.
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CALENDAR OF FUTURE MEETINGS

Forty-second Annual Meeting, University of Pennsylvania, Philadelphia, Pennsylvania, January 22–23, 1959.

Fortieth Summer Meeting, University of Utah, Salt Lake City, Utah, August 31–September 3, 1959.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, University of Pittsburgh, May 2, 1959.
ILLINOIS, Millikin University, Decatur, May 8–9, 1959.
INDIANA, Valparaiso University, May 2, 1959.
IOWA, Iowa Wesleyan University, Mount Pleasant, April 17, 1959.
KANSAS, Marymount College, Salina, April 11, 1959.
KENTUCKY, Centre College of Kentucky, Danville, April, 1959.
LOUISIANA-MISSISSIPPI, Buena Vista Hotel, Biloxi, Mississippi, February 13–14, 1959.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK, Polytechnic Institute of Brooklyn, April 18, 1959.
MICHIGAN, Michigan State University of Agriculture and Applied Science, East Lansing, March 28, 1959.
MINNESOTA, University of Minnesota, Minneapolis, April 25, 1959.
MISSOURI, Lindenwood College, St. Charles, April 25, 1959.
NEBRASKA, University of Nebraska, Lincoln, April 18, 1959.
NEW JERSEY
NORTHEASTERN
NORTHERN CALIFORNIA, Stanford University, January 17, 1959.
OHIO, Miami University, Oxford, May 9, 1959.
OKLAHOMA, Tulsa University, Tulsa, Oklahoma, Spring, 1959.
PACIFIC NORTHWEST, University of Oregon, Eugene, June 19, 1959.
PHILADELPHIA
ROCKY MOUNTAIN, Utah State University of Agriculture and Applied Science, Logan, May 8–9, 1959.
SOUTHEASTERN, East Tennessee State College, Johnson City, March 20–21, 1959.
SOUTHERN CALIFORNIA, University of Redlands, March 14, 1959.
SOUTHWESTERN, Arizona State College, Tempe, Spring, 1959.
TEXAS, University of Texas, Austin, April, 1959.
UPPER NEW YORK STATE, Hartwick College, Oneonta, May 9, 1959.
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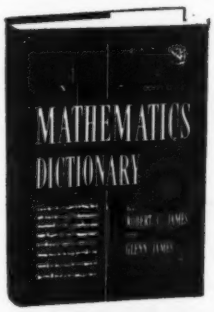

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